Applications of First Order Differential Equations (Sections 3.2, 3.3)

The initial value problem

$$\frac{dx}{dt} = kx, \quad x(t_0) = x_0$$

where *k* is a constant of proportionality, occurs in modeling physical phenomenon involving growth and decay. This is a 1^{st} order linear IVP.

When $k > 0$, then the above IVP can model population growth. But, as we've seen from the solution, the population would exhibit unbounded exponential growth. This is completely unrealistic in most situations.

P.H. Verhulst was interested in modeling human population growth. One of the equations he studied was

$$\frac{dP}{dt} = P(a - bP)$$

where *a* and *b* are positive constants. This equation is known as the **logistic equation**, and its solution curve is called the **logistic curve**. It is a 1^{st} order nonlinear ODE.

If *a* is the average birth rate, let's assume that the average death rate is proportional to the population $P(t)$ at time *t*, call it *bP*. Thus, if $\frac{1}{P}\frac{dP}{dt}$ represents the rate of growth per individual in a population, then

$$\frac{1}{P}\frac{dP}{dt} = a - bP$$

and so

$$\frac{dP}{dt} = P(a - bP)$$

We shall see that the solution of this equation is bounded as $t \to \infty$.

To solve, separate the variables.

$$\frac{dP}{dt} = P\left(a - bP\right) \quad \Rightarrow \quad \frac{dP}{P\left(a - bP\right)} = dt$$

Note that $\frac{1}{P(a-bP)} = \frac{1/a}{P} + \frac{b/a}{a-bP}$. Thus, we have the following.

$$\begin{aligned}
\int \frac{dP}{P\left(a-bP\right)} = \int dt \quad & \Rightarrow \quad \int \frac{1/a}{P}\,dP + \int \frac{b/a}{a-bP}\,dP = \int dt \\
& \Rightarrow \quad \frac{1}{a}\ln\left|P\right| - \frac{1}{a}\ln\left|a-bP\right| = t + C \\
& \Rightarrow \quad \ln\left|\frac{P}{a-bP}\right| = at + aC \\
& \Rightarrow \quad \frac{P}{a-bP} = c_1 e^{at} \\
& \Rightarrow \quad P = c_1 e^{at}\left(a - bP\right) \\
& \Rightarrow \quad P = c_1 a e^{at} - c_1 b e^{at} P \\
& \Rightarrow \quad P + c_1 b e^{at} P = c_1 a e^{at} \\
& \Rightarrow \quad \left(1 + c_1 b e^{at}\right) P = c_1 a e^{at} \\
& \Rightarrow \quad P = \frac{c_1 a e^{at}}{1 + c_1 b e^{at}} \\
& \Rightarrow \quad P = \frac{c_1 a}{e^{-at} + c_1 b}
\end{aligned}$$

If we now apply the initial condition $P(0) = P_0$, then we have the following.

$$\begin{aligned}
P(0) = \frac{c_1 a}{1 + c_1 b} = P_0 \quad & \Rightarrow \quad c_1 a = P_0\left(1 + c_1 b\right) \\
& \Rightarrow \quad c_1 a = P_0 + c_1 b P_0 \\
& \Rightarrow \quad c_1 a - c_1 b P_0 = P_0 \\
& \Rightarrow \quad c_1\left(a - bP\right) = P_0 \\
& \Rightarrow \quad c_1 = \frac{P_0}{a - bP}
\end{aligned}$$

Thus, we obtain the following.

$$P(t) = \frac{aP_0}{bP_0 + \left(a - bP_0\right)e^{-at}}$$