Reduction of Order (Section 4.2)

Suppose that y_1 is a solution of $\dfrac{d^2y}{dx^2} + p(x)\dfrac{dy}{dx} + q(x)y = 0$. Our objective is to find a y_2 that is linearly independent of y_1. Thus, $y_2 = u(x)y_1$, that is, y_2 is not a constant multiple of y_1.

Substituting $y_2 = u(x)y_1$ into the original equation, we get the following.

$$
\begin{aligned}
&y_2' = u'y_1 + uy_1' \\
&y_2'' = u''y_1 + u'y_1' + u'y_1' + uy_1'' = u''y_1 + 2u'y_1' + uy_1'' \\
&y'' + p(x)y' + q(x)y = \left[u''y_1 + 2u'y_1' + uy_1''\right] + p(x)\left[u'y_1 + uy_1'\right] + q(x)uy_1 = 0
\end{aligned}
$$

Putting together the u, u', and u'', we have the following.

$$\underbrace{\left[uy_1'' + p(x)uy_1' + q(x)uy_1\right]}_{=0 \text{ since } y_1 \text{ is a solution}} + \left[2u'y_1' + p(x)u'y_1\right] + u''y_1 = 0$$

Let $w = u'$. Thus, $w = u''$. Making this substitution, we now have the following.

$$w'y_1 + (2y_1' + p(x)y_1)w = 0$$

This equation is a linear 1[st] order equation in w. We solve by separation of variables.

Ultimately, we end up with $u = \displaystyle\int \frac{e^{-\int p(x)dx}}{y_1^2}dx + c$. Choosing $c = 0$, we find that a second solution of $\dfrac{d^2y}{dx^2} + p(x)\dfrac{dy}{dx} + q(x)y = 0$ is $y_2 = y_1\displaystyle\int \frac{e^{-\int p(x)dx}}{y_1^2}dx$.