Summary of Important Theoretical Results

***Theorem:* Existence and Uniqueness of a Solution to an IVP**
Let $a_n(x)$, $a_{n-1}(x)$, …, $a_2(x)$, $a_1(x)$, $a_0(x)$ and $g(x)$ be continuous on the interval *I* and let $a_n(x) \neq 0$ for every *x* in the interval. If $x = x_0$ is any point in this interval, then a solution $y(x)$ of the above initial value problem exists on the interval and is unique.

Theorem:* Criterion for Linearly Independent *Functions
Let f_1, f_2, …, f_n possess at least $n - 1$ derivatives on an interval *I*. If $W(f_1, f_2, ..., f_n) \neq 0$ for at least one point in the interval *I*, then the set of functions is linearly independent. ■

Equivalent Statement: Let f_1, f_2, …, f_n possess at least $n - 1$ derivatives on an interval *I*. If the set of functions is linearly dependent, then $W(f_1, f_2, ..., f_n) = 0$ for all $x \in I$.

***Theorem:* Principle of Superposition – Homogeneous Equations**
Let y_1, y_2, …, y_k be linearly independent solutions of the homogeneous n^{th} order ODE on an interval *I*. Then, the linear combination $y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_k y_k(x)$ is also a solution on the interval *I*, where c_i are arbitrary constants. ■

Corollary:
(*i*) A constant multiple $y = c_1 y_1(x)$ of a solution $y_1(x)$ of a homogeneous linear DE is also a solution.
(*ii*) A homogeneous linear DE always possesses the trivial solution $y = 0$. ■

Theorem:* Criterion for Linearly Independent *Solutions
Let y_1, y_2, …, y_n be solutions of linear n^{th} order homogeneous DE. Then, the set of solutions is linearly independent on *I* if and only if $W(y_1, y_2, ..., y_n) \neq 0$ for every *x* in *I*. ■

Equivalent Statements: Let y_1, y_2, …, y_n be solutions of linear n^{th} order homogeneous DE.

- If the set of solutions is linearly independent on *I*, then $W(y_1, y_2, ..., y_n) \neq 0$ for every *x* in *I*.
- If $W(y_1, y_2, ..., y_n) \neq 0$ for every *x* in *I*, then the set of solutions is linearly independent on *I*.
- If $W(y_1, y_2, ..., y_n) = 0$ for at least one *x* in *I*, then the set of solutions is linearly dependent on *I*.
- If the set of solutions is linearly dependent on *I*, then $W(y_1, y_2, ..., y_n) = 0$ for at least one *x* in *I*.

If the set of solutions is linearly independent on *I*, then $W(y_1, y_2, ..., y_n) = 0$ for at least one *x* in *I*.

<u>Theorem:</u> Existence of a Fundamental Solution Set

There exists a fundamental solution set of solutions for the linear n^{th} order homogeneous DE on the interval *I*. ■

<u>Theorem:</u> General Solution – Homogeneous Equations

Let $y_1,\ y_2,\ \ldots,\ y_n$ be a fundamental solution set of the linear n^{th} order homogeneous DE on the interval *I*. Then, the **general solution** of the equation on the interval is $y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x)$. ■

<u>Theorem:</u> General Solution – Nonhomogenous Equations

Let y_p be a particular solution of

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x),$$

and let $y_1,\ y_2,\ \ldots,\ y_n$ be a fundamental solution set of the associated homogenous DE

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_2(x)\frac{d^2 y}{dx^2} + a_1(x)\frac{dy}{dx} + a_0(x)y = 0$$

on the interval *I*. Then, the general solution of the nonhomogeneous DE is

is $y = c_1 y_1(x) + c_2 y_2(x) + \cdots + c_n y_n(x) + y_p$. ■