The Laplace Transform

In Calculus, we learned that differentiation and integration are *transforms*, i.e., these operations transform a function into another function. Both of these transforms possess a linearity property.

$$\frac{d}{dx}[\alpha f(x) + \beta g(x)] = \alpha \frac{d}{dx} f(x) + \beta \frac{d}{dx} g(x)$$
$$\int [\alpha f(x) + \beta g(x)]dx = \alpha \int f(x)dx + \beta \int g(x)dx$$

In words, we say that the transform of a linear combination of functions is a linear combination of the transforms.

We are interested in a special integral transform. To begin our discussion of this special transform, consider $\int_a^b K(s,t) f(t)dt$. This transform changes the function *f* of the variable *t* into a new function *F* of the variable *s*. The function $K(s,t)$ is sometimes called the **kernel** of the transformation.

We are interested in transforms of the following kind: $\int_0^\infty K(s,t) f(t)dt$. (Recall that $\int_0^\infty K(s,t) f(t)dt = \lim_{b\to\infty} \int_0^b K(s,t) f(t)dt$.) Of particular interest is the transform with kernel $K(s,t) = e^{-st}$. This choice gives us an important integral transform. Formally, we have the following.

Let *f* be a function defined for $t \geq 0$. Then,

$$L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$$

is called the **Laplace transform** of the function *f*, provided that the function converges. We often write $L\{f(t)\} = F(s)$, since this transform changes the function *f* of the variable *t* into a new function of the variable *s*.

The following is a list of some common Laplace transforms.

<u>Theorem:</u> Let *f* be a function defined for $t \geq 0$, and let *s* be suitably restricted so that $L\{f(t)\} = \int_0^\infty e^{-st} f(t)dt$ converges. Then, the following are the transforms of some basic functions.

$$L\{1\} = \frac{1}{s} \qquad L\{t^n\} = \frac{n!}{s^{n+1}},\ n = 1, 2, 3, \ldots \qquad L\{e^{at}\} = \frac{1}{s-a}$$

$$L\{\sin kt\} = \frac{k}{s^2+k^2} \qquad L\{\cos kt\} = \frac{s}{s^2+k^2}$$

$$L\{\sinh kt\} = \frac{k}{s^2-k^2} \qquad L\{\cosh kt\} = \frac{s}{s^2-k^2}$$

The Laplace transform is a linear transform. That is,

$$l\{\alpha f(t) + \beta g(t)\} = \alpha l\{f(t)\} + \beta l\{g(t)\}.$$

There are a few questions about the Laplace transform that we should answer before exploring it any further. In particular, when does the Laplace transform of a given function exist and when is it unique? The following theorems and definitions address these issues.

A function $f(t)$ is **continuous** on *I* if $\lim_{t\to a} f(t) = f(a)\ \forall\ a \in I$.
A function $f(t)$ is **piecewise continuous** on $a \leq t \leq b$ if [*a*, *b*] can be divided into finitely many subintervals so that

a) $f(t)$ is continuous in the interior of each subinterval.
b) $f(t)$ has a finite limit as *t* approaches each endpoint of the subinterval.

$\int_a^b g(t)dt$ exists if $g(t)$ is piecewise continuous on [*a*, *b*].

$\int_0^b e^{-st} f(t)dt$ exists if $f(t)$ is continuous $\forall\ t \geq 0$ and if $b < +\infty$.

In order for $\lim_{b\to+\infty} \int_0^b e^{-st} f(t)dt$ to exist, we must limit the growth of $f(t)$ as $t \to +\infty$.

A function *f* is of **exponential order** if there exists nonnegative constants *M*, *c*, and *T* such that $|f(t)| \leq Me^{ct}$, $t > T$.

<u>Theorem:</u> *The Existence of the Laplace Transform*
If *f* is piecewise continuous for $t \geq 0$ and of exponential order for *t* > *T*, then $L\{f(t)\} = F(s)$ exists $\forall\ s > c$.

Pf: See text, page 284.

Throughout our discussions of the Laplace transform, we shall only be interested in piecewise continuous functions of exponential order.

Theorem: *Uniqueness of the Laplace Transform*
Suppose that *f* and *g* satisfy the hypotheses of the existence theorem so that their Laplace transforms $F(s)$ and $G(s)$ exist. If $F(s) = G(s) \;\; \forall \;\; s > c$ for some *c*, then $f(t) = g(t)$ where *f* and *g* are continuous on $[0, +\infty)$.

Okay, great. We now know that the Laplace transform exists and is unique. What does this tell us? Well, since the Laplace transform exists and is unique, we know that no two different functions can have the same Laplace transform. Thus, if *F*(*s*) is the transform of a continuous function *f*(*t*), then *f*(*t*) is uniquely determined. If $F(s) = L\{f(t)\}$, then $f(t)$ is called the **inverse Laplace transform** of *F*(*s*), and $f(t) = L^{-1}\{f(s)\}$.

The following is a list of common inverse Laplace transforms.

$$1 = L^{-1}\left\{\frac{1}{s}\right\} \qquad t^n = L^{-1}\left\{\frac{n!}{s^{n+1}}\right\} \qquad e^{at} = L^{-1}\left\{\frac{1}{s-a}\right\}$$

$$\sin kt = L^{-1}\left\{\frac{k}{s^2+k^2}\right\} \qquad \cos kt = L^{-1}\left\{\frac{s}{s^2+k^2}\right\}$$

$$\sinh kt = L^{-1}\left\{\frac{k}{s^2-k^2}\right\} \qquad \cosh kt = L^{-1}\left\{\frac{s}{s^2-k^2}\right\}$$