Classification of Differential Equations (Section 1.1)

Collectively, the words *differential* and *equation* suggest some kind of equation that contains derivatives. Indeed, this is the case. The primary focus of this course is solving equations that contain derivatives. But, before we start solving anything, we need some basic definitions and terminology.

In Calculus, you learned that the given a function $y = f(x)$, the derivative

$$\frac{dy}{dx} = f'(x)$$

is itself a function of x and can be found by some appropriate rule (e.g.: the power rule, the product rule, the chain rule, etc.). The problem we face in this course is not this: given a function $y = f(x)$, find its derivative. Rather, our problem is this: if given the equation $\frac{dy}{dx} = g(x)$, we must somehow find the function $y = f(x)$ that satisfies the equation. That is, we need to solve the equation.

A **differential equation** is an equation containing the derivatives or differentials of one or more dependent variables, with respect to one or more independent variables.

Example 1: The equation $\frac{dy}{dx} = e^x$ is an example of a differential equation.

Before we begin solving differential equations, let's classify them. We generally classify differential equations based on order, linearity, and type.

Order

The order of a differential equation is the order of the highest derivative in the equation.

Example 2: $\frac{dy}{dx} = e^x$ is a 1st order differential equation.

Example 3: $\frac{d^2y}{dx^2} + 4\frac{dy}{dx} - 5y = 0$ is a 2nd order differential equation.

Example 4: $x^2\frac{d^3y}{dx^3} + 5x\frac{dy}{dx} + 2y = e^x$ is a 3rd order differential equation.

Symbolically, we express an n^{th} order ordinary differential equation in one dependent variable by the **general form**

$$F(x, y, y', \ldots, y^{(n)}) = 0.$$

For example, a 1st order differential equation is symbolically represented as $F(x, y, y') = 0$, and a 2nd order differential equation is symbolically represented as $F(x, y, y', y'') = 0$

We can express an n^{th} order ordinary differential equation in one dependent variable by the **normal form**

$$\frac{d^n y}{dx^n} = F(x, y, y', \ldots, y^{(n-1)}).$$

In normal form, we place the highest derivative on one side of the equation and every thing else on the other side.

$4xy' + y - x = 0$	General form of a 1st order differential equation
$y' = \dfrac{x - y}{4x}$	Normal form of a 1st order differential equation

Linearity

A differential equation is said to be **linear** if

1) the dependent variable and all of its derivatives or differentials are of first degree (that is, the power of the dependent variable and all of its derivatives is 1),

2) the coefficient of each term depends at most on the independent variable, and

3) neither the dependent variable nor any of its derivatives is the argument of some other function.

A differential equation that is not linear is called **nonlinear**.

Example 5: $y'' - 2y' + y = 0$ is a linear differential equation.

Example 6: $x^2 \dfrac{d^3 y}{dx^3} + 5x \dfrac{dy}{dx} + 2y = e^x$ is a linear differential equation.

Example 7: $\dfrac{d^2 y}{dx^2} - 3x\left(\dfrac{dy}{dx}\right)^3 + y = 0$ is a nonlinear differential equation.

Example 8: $y\frac{dy}{dx} + y = 0$ is a nonlinear differential equation.

Example 9: $\frac{d^2y}{dx^2} = \cos(y)$ is a nonlinear differential equation.

Type

Differential equations come in two types: ordinary differential equations and partial differential equations.

A differential equation that contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable is called an **ordinary differential equation (ODE)**.

A differential equation that contains partial derivatives of one or more dependent variables of two or more independent variables is called a **partial differential equation (PDE)**.

Derivatives are written using the following notations.

- **Liebniz notation** – $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \ldots, \frac{d^ny}{dx^n}$ …. Clearly displays both dependent and independent variable.
- **Prime notation** – $y', y'', y''', \ldots, y^{(n)}, \ldots$
- **Newton's dot notation** – $\dot{s}, \ddot{s}, \dddot{s}$. Used to represent derivatives with respect to time.
- **Subscript notation** – u_{xx}, u_{tt}. Represent partial derivatives.

Example 10: Classify $B'(t) = kB(t)$.

This is the growth equation. It is a *1st order* differential equation because the highest derivative present is the first derivative. It is *linear* because (1) the dependent variable and all of its derivatives are of the first degree, (2) the coefficient of each term depends on at most the independent variable, *t*, and (3) neither the dependent variable nor any of its derivatives is the argument of some other function. This DE is *ordinary* because it involves ordinary derivatives. We can therefore refer to this DE as a *1st order linear ordinary differential equation*.

Example 11: $P'(t) = P(t)\left[a - bP(t)\right]$.

This equation is known as the *logistic equation*. It provides a slightly better model for population growth. It is a *1st order* differential equation because the highest derivative is the first derivative. It is *nonlinear* because the dependent variable P is of the second degree. This DE is *ordinary* because it involves ordinary derivatives. We can therefore refer to this DE as a *1st order nonlinear ordinary differential equation*.

Example 12: Classify $LQ''(t) + RQ'(t) + \frac{1}{C}Q(t) = E(t)$.

This is *Kirchoff's Law*. It is important in circuit analysis. It is a *2nd order* differential equation because the highest derivative is a second derivative. It is *linear* because (1) the dependent variable and all of its derivatives are of the first degree, (2) the coefficient of each term depends on at most the independent variable, t, and (3) neither the dependent variable nor any of its derivatives is the argument of some other function. This DE is *ordinary* because it involves ordinary derivatives. We can therefore refer to this DE as a *2nd order linear ordinary differential equation*.

Example 13: Classify $\frac{\partial^2 u}{dx^2}(x,t) = \frac{\partial^2 u}{dt^2}(x,t)$.

Originally used to study heat flow, the *heat equation* has applications to the study of options pricing in mathematical finance. It is a *2nd order* differential equation because the highest derivative present is a second derivative. It is *linear* because (1) the dependent variable and all of its derivatives are of the first degree, (2) the coefficient of each term depends on at most the independent variables, and (3) neither the dependent variable nor any of its derivatives is the argument of some other function. This DE is *partial* because it involves partial derivatives. We can therefore refer to this DE as a *2nd order linear partial differential equation*.

Example 14: Classify $\theta'' + \frac{g}{l}\sin\theta = 0$.

This equation is known as the *pendulum equation*. It models the motion of a simple pendulum suspended on a rigid rod. It is a *2nd order* differential equation because the highest derivative present is a second derivative. It is *nonlinear* because the dependent variable θ is the argument of some other function (in this case, the sine function). This DE is *ordinary* because it involves ordinary derivatives. We can therefore refer to this DE as a *2nd order nonlinear ordinary differential equation*.

Other Examples

Example 15: Classify $4xy' + y - x = 0$.

It is a *1st order* differential equation because the highest derivative present is the first derivative. It is *linear* because (1) the dependent variable and all of its derivatives are of the first degree, (2) the coefficient of each term depends on at most the independent variable, *x*, and (3) neither the dependent variable nor any of its derivatives is the argument of some other function. This DE is *ordinary* because it involves ordinary derivatives. We can therefore refer to this DE as a *1st order linear ordinary differential equation*.

Example 16: Classify $y'' + y = 0$.

It is a *2nd order* differential equation because the highest derivative present is the second derivative. It is *linear* because (1) the dependent variable and all of its derivatives are of the first degree, (2) the coefficient of each term depends on at most the independent variable, and (3) neither the dependent variable nor any of its derivatives is the argument of some other function. This DE is *ordinary* because it involves ordinary derivatives. We can therefore refer to this DE as a *2nd order linear ordinary differential equation*.

Example 17: Classify $(1-y)\dfrac{d^3y}{dx^3} + 2y = 0$.

It is a *3rd order* differential equation because the highest derivative present is the third derivative. It is *nonlinear* because the coefficient of $\dfrac{d^3y}{dx^3}$ depends on more than just the independent variable *x*; the coefficient of $\dfrac{d^3y}{dx^3}$ is $1-y$. This DE is *ordinary* because it involves ordinary derivatives. We can therefore refer to this DE as a *3rd order nonlinear ordinary differential equation*.

Example 18: Classify $\dfrac{d^4y}{dx^4} + 2\dfrac{d^3y}{dx^3} - \left(\dfrac{dy}{dx}\right)^2 + y = 0$.

It is a *4th order* differential equation because the highest derivative present is the fourth derivative. It is *nonlinear* because the $\dfrac{dy}{dx}$ term is squared. This DE is *ordinary* because it involves ordinary derivatives. We can therefore refer to this DE as a *4th order nonlinear ordinary differential equation*.

Example 19: Classify $a^2 \frac{\partial^2 u}{dx^2}(x,t) = \frac{\partial^2 u}{dt^2}(x,t)$.

This is the *wave equation*. It models behavior such as the motion of a plucked guitar string. It is a *2nd order* differential equation because the highest derivative present is a second derivative. It is *linear* because (1) the dependent variable and all of its derivatives are of the first degree, (2) the coefficient of each term depends on at most the independent variables, and (3) neither the dependent variable nor any of its derivatives is the argument of some other function. This DE is *partial* because it involves partial derivatives. We can therefore refer to this DE as a *2nd order linear partial differential equation*.

Primitives (optional)

Starting with the relation

$$y = a \sin x \qquad (1)$$

where a is an arbitrary constant, we obtain the following DE by differentiation.

$$y' = a \cos x. \qquad (2)$$

From the original relation, we see that $a = \frac{y}{\sin x}$. Substituting this into our differential equation gives us, after some manipulation, the following.

$$y' - y \cot x = 0. \qquad (3)$$

Thus, (3) is a differential equation that has (1) as its solution.

But this is not the only differential equation that can have (1) as its solution. Differentiating (1) twice, we have

$$y'' = -a \sin x. \qquad (4)$$

Once again, using the fact that $a = \frac{y}{\sin x}$, we obtain the following.

$$y'' + y = 0. \qquad (5)$$

Thus, (5) is also a differential equation that has (1) as its solution. We can come up with other equations that has (1) as its solution. While this is true, only one of these equation has (1) as its *primitive*. In particular, we say that (3) is the differential equation that has (1) as its primitive. Whenever we consider the primitive of a differential equation, we consider the differential equation of lowest order that has the given relation has its solution. A primitive involving one arbitrary constant gives rise to a 1st order DE. Likewise, a primitive involving two arbitrary constants gives rise to a 2nd order DE. In general, a primitive involving two arbitrary constants gives rise to an nth order DE.

Let's look at an example of a primitive involving two arbitrary constants. Consider the following.

$$y = a\cos x + b\sin x \qquad (6)$$

where *a* and *b* are arbitrary constants. Upon differentiating, we have

$$y' = -a\sin x + b\cos x \qquad (7)$$

In order to eliminate *a* and *b*, three relations involving them are required. To obtain a third one, we'll differentiate (7).

$$y'' = -a\cos x - b\sin x \qquad (8)$$

We can now eliminate *a* and *b*. (Do this as an exercise). When we do so, we obtain the following.

$$y'' + y = 0 \qquad (9)$$

This equation has (6) as its solution. Here, we say that (9) is the DE having (6) as its primitive.

The term primitive is an older term that, to the best of my knowledge, is not used all that often in more modern texts on elementary differential equations. This is perhaps due to the fact that, in thinking about primitives and differential equations, the primitive comes first and the differential equation comes next. In what we do in this class, the differential equation comes first and the solution (its primitive) comes next.