Solutions of Differential Equations (Section 1.1)

Once again, our primary goal in this course is to solve differential equations. Thus, we need to define what we mean by a *solution*.

A function f defined on some interval I, that when substituted into a differential equation reduces the equation to an identity, is said to be a **solution** of the equation on the given interval.

A solution of an n^{th} order ordinary differential equation $F(x, y, y', ..., y^n) = 0$ is a function f that possesses at least n derivatives and satisfies the equation $F(x, f(x), f'(x), ..., f^{(n)}(x)) = 0$ for all x in I.

The interval I can represent an open interval (a, b), a closed interval $[a, b]$, an infinite interval $(-\infty, \infty)$.

<u>Example 1:</u> Verify that $y = e^{-x/2}$ is a solution of the differential equation $2y' + y = 0$.

<u>Example 2:</u> Verify that $y = xe^x$ is a solution of the differential equation $y'' - 2y' + y = 0$.

In thinking about solutions, we should think about the interval on which that solution is defined. The interval I is called the **interval of definition**, the **interval of existence**, the **interval of validity**, or the **domain** of the solution.

<u>Example 3:</u> If C is a constant and $y(x) = \frac{1}{C - x}$, then $\frac{dy}{dx} = \frac{1}{(C - x)^2} = y^2$. Thus

$$y(x) = \frac{1}{C - x}$$

defines a solution of the differential equation

$$\frac{dy}{dx} = y^2$$

(which, by the way, is a 1^{st} order nonlinear ordinary differential equation) on any interval not containing $x = C$. We say that $y(x) = \frac{1}{C - x}$ defines a solution of $\frac{dy}{dx} = y^2$. Note there is one solution for each value of the parameter C. When $C = 1$, we have the *particular solution* $y(x) = \frac{1}{1 - x}$. Note that $x = 1$ is a vertical asymptote. Thus, the formula $y(x) = \frac{1}{1-x}$ actually defines two solutions, one over the interval $(-\infty, 1)$, and another over the interval $(1, \infty)$. ■

The graph of the solution *f* of an ODE is called a **solution curve**. The graph of the function *f* may be different than the graph of the solution *f*. This is certainly the case in example above.

When solving a 1st order linear ODE of the general form $F(x, y, y') = 0$, we usually obtain a solution containing a single arbitrary constant or parameter *C*. Intuitively, this makes sense. After all, we have an equation containing a single derivative. If we wish to solve that equation, it stands to reason that we'll have to integrate once. When we calculate an indefinite integral, we always have a constant of integration.

A solution of $F(x, y, y') = 0$ containing one arbitrary constant is called a **one-parameter family of solutions**.

A solution of $F(x, y, y', y'') = 0$ containing two arbitrary constants and is called a **two-parameter family of solutions**.

A solution of $F(x, y, y', \ldots, y^{(n)}) = 0$ contains *n* arbitrary constants and is called an ***n*-parameter family of solutions**.

A solution of a DE that is free of arbitrary parameters is called a **particular solution**.

When a solution cannot be obtained by specifying the parameters in the family of solutions, it is called a **singular solution**.

If every solution of the differential equation can be obtained by from an *n*-parameter family of solutions, then the *n*-parameter family of solutions is called a **general solution**.

Example 4: $y = \left(\frac{1}{4}x^2 + c\right)^2$ is a *one-parameter family of solutions* of the ODE $y' = xy^{1/2}$. (Verify this.) When $c = 0$, the resulting *particular solution* is $y = \frac{1}{16}x^4$. Note that $y = 0$ is also a solution (check this), but it cannot be obtained by specifying a value for *c* in the one-parameter family of solutions. Thus, $y = 0$ is a singular solution, and, as such, $y = \left(\frac{1}{4}x^2 + c\right)^2$ cannot be referred to as a general solution. ■

The solution $y = 0$ is called the **trivial solution**.

Example 5: $B(t) = Ce^{2t}$ is a *one-parameter family of solutions* of the ODE $B'(t) = 2B(t)$. (Verify this.) When $c = 10$, we the resulting *particular solution* is $B(t) = 10e^{2t}$. Note that $B(t) = 0$ is also a solution (the trivial solution), and (unlike in the previous example) can be obtained by specifying a values for C in the one-parameter family of solutions. Every solution to the equation $B'(t) = 2B(t)$ can be obtained from the solution $B(t) = Ce^{2t}$ by simply specifying the constant C. Hence, there are no singular solutions, and so $B(t) = 10e^{2t}$ is a *general solution*. ■

A solution in which the dependent variable is expressed solely in terms of the independent variable and constants is called an **explicit solution**.

Sometimes, we encounter solutions that are not explicit. A relation $G(x, y) = 0$ is called an **implicit solution** of an ODE on an interval *I* provided it defines one or more explicit solutions on *I*.

Example 6: The relation $x^2 + y^2 - 4 = 0$ is an implicit solution of $\frac{dy}{dx} = -\frac{x}{y}$ on the interval (−2, 2). (Verify this.) This relation defines two explicit functions on this interval: $y = \sqrt{4 - x^2}$ and $y = -\sqrt{4 - x^2}$. ■