Differential Equations and Mathematical Models (Section 1.2)

The study of differential has three principal goals.

1) To discover the DE that describes a particular phenomena.
2) To find the solution (exactly or approximately) of a particular DE.
3) To interpret the solution.

In algebra, our primary objective is to find the numbers that satisfy a particular equation. In differential equations, we seek the functions that satisfy a particular equation. We want to find all of the solutions, if possible.

Once again, in this course, our primary focus is on objective 2: to find the solution (exactly or approximately) of a particular differential equation. We will not try to discover the DE that describes a particular phenomena. This is the task of mathematical modeling.

Mathematical Models

A **mathematical model** consists of variables that describe a given situation together with one or more equations relating these variables that are known or assumed to hold.

The following examples illustrate some mathematical models.

Example 1: *Newton's Law of Cooling* The time rate of change (the rate of change with respect to time) of the temperature $T(t)$ of a body is proportional to the difference between the temperature T of the body and the temperature A of the surrounding medium.

$$\frac{dT}{dt} = -k(T - A)$$

where $k > 0$. If $T > A$ (that is, if the temperature of the body is more than the temperature of the surrounding medium), then the rate of change of the temperature of a body is negative, that is, it's cooling off. If $T < A$ (that is, if the temperature of the surrounding medium is more than the temperature of the body), then the rate of change of the temperature of the body is positive, that is, it's heating up. ■

Example 2: *Population Growth* The time rate of change of a population $P(t)$ with constant births and deaths is (in simple cases) proportional to the size of the population.

$$\frac{dP}{dt} = kP$$

where k is a constant. Each function of the form

$$P(t) = Ce^{kt}$$

is a solution for all real numbers t. (Verify this.) So, even if the constant k is known, there are infinitely many different solutions, one for each choice of the constant C.

Sometimes we can find a *particular solution*. Suppose that $P(t) = Ce^{kt}$ describes the population of a colony of bacteria. Suppose that at $t = 0$ (hours), the population is 2000. After an hour, suppose that the population doubles. Thus,

$$\begin{aligned} P(0) &= 2000 \Rightarrow Ce^0 = 2000 \Rightarrow C = 2000 \\ P(1) &= 4000 \Rightarrow Ce^k = 4000 \end{aligned}$$

From this, it follows that

$$\begin{aligned} 2000e^k &= 4000 \\ e^k &= 2 \\ \ln(e^k) &= \ln 2 \\ k &= \ln 2 \end{aligned}$$

Now that we know C and k, we can write down a formula for $P(t)$.

$$\begin{aligned} P(t) &= 2000e^{t\ln 2} \\ &= 2000(e^{\ln 2})^t \\ &= 2000 \cdot 2^t \end{aligned}$$

We can use this solution to predict the population for other time values. ■

In Example 2, $P(0) = 2000$ is called an **initial condition** because $t = 0$ is the starting time.

The differential equation and its solution in Example 2 certainly do not adequately describe the real world. There are no choices of the constants k and C that actually describes the growth of the human population. This should not be regarded as a failure of the model. Rather, it points to the fact that we need a more sophisticated model to describe something as complex as human population growth. Certainly, we would need a model that takes into account births, deaths, food supply, climate changes, etc.

A mathematical model is subject to two seemingly contradictory requirements: it must be sufficiently detailed so as to represent the real-world problem accurately, and it must be sufficiently simple so that it can be mathematically analyzed. If the model is too complicated, the mathematical analysis may be too difficult to carry out. If the model is too simple, then the results may not relate to the real-world problem.