First Order Differential Equations: Preliminary Theory (Section 2.1)

Initial Value Problem (IVP)

A (first order) differential equation $\frac{dy}{dx} = f(x, y)$ in conjunction with a specified side condition $y(x_0) = y_0$ is called an **initial value problem**. The side condition is called an **initial condition**. We seek a *particular solution* (a solution free of arbitrary parameters) defined on some interval *I* that passes through the point (x_0, y_0).

We saw an example of a first order IVP earlier. Recall, we had the following.

$$\frac{dP}{dt} = kP, \quad P(0) = 2000$$

Existence and Uniqueness of a Solution to an IVP

When dealing with an initial-value problem, two fundamental questions arise.

> Does a solution exist?
> If so, is the solution unique (i.e., is there precisely one solution)?

Sometimes, the answer to the second question is no.

Example 1: The IVP $y' = xy^{1/2}$, $y(0) = 0$ has two solutions: $y = \frac{1}{16}x^4$ and $y = 0$. (See Example 4 from the second set of notes.) The graphs of both functions pass through the point (0, 0). Thus, there is not a unique solution. ■

The following theorem is sufficient to guarantee that a solution exists and that it is unique.

Theorem: Let *R* be a rectangular region in the *xy*-plane defined by $a \le x \le b$, $c \le y \le d$ that contains the point (x_0, y_0) in its interior. If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous on *R*, then there exists an interval *I* centered at x_0 and a unique function $y(x)$ defined on *I* satisfying the initial value problem $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$. ■

Example 2: Determine a region of the *xy*-plane for which $x\frac{dy}{dx} = y$ would have a unique solution through a point (x_0, y_0) in the region.

Here, $f(x,y) = \frac{y}{x}$ and $\frac{\partial f}{\partial y} = \frac{1}{x}$. These are continuous anywhere that $x \neq 0$. Thus, by our theorem, the DE has a unique solution in any region where $x \neq 0$.