Separable Equations (Section 2.2)

We'll begin our investigation of solution methods by looking at some 1st order ODEs.

Solution by Integration

We begin with the simplest of all differential equations.

$$\frac{dy}{dx} = g(x)$$

Note that the left hand side is free of the variable *y*. Note that this is a 1st order ODE. It may be linear or nonlinear. Simply integrate both sides with respect to *x*.

Example 1: Solve $\frac{dy}{dx} = 1 + e^{2x}$.

Separable Equations

The previous example above is a special case of a separable equation. A DE of the form

$$\frac{dy}{dx} = g(x)h(y)$$

is said to be **separable** or to have **separable variables**. Note that this is a 1st order ODE. It may be linear or nonlinear.

We can solve such an equation as follows.

$\frac{dy}{dx} = g(x)h(y)$	Original equation.
$\frac{1}{h(y)}\frac{dy}{dx} = \frac{g(x)h(y)}{h(y)}$	Divide both sides by *h*(*y*).
$p(y)\frac{dy}{dx} = g(x)$	Let $p(y) = \frac{1}{h(y)}$.

Let $y = \phi(x)$ be a solution to the DE. Then,

$p(\phi(x))\phi'(x) = g(x)$	$y = \phi(x)$ and $\frac{dy}{dx} = \phi'(x)$.
$\int p(\phi(x))\phi'(x)dx = \int g(x)dx$	Integrate both sides with respect to *x*.
$\int p(y)dy = \int g(x)dx$	$y = \phi(x)$ and $dy = \phi'(x)dx$.

$H(y) + c_1 = G(x) + c_2$ — $H(y)$ is the antiderivative of $p(y) = \frac{1}{h(y)}$ and $G(x)$ is the antiderivative of $g(x)$.

$H(y) = G(x) + c$ — $c = c_2 - c_1$.

Example 2: Solve $\frac{dy}{dx} = y^2 - 4$.

Example 3: Solve $\frac{dy}{dx} = y^2 - 4$.

Example 4: Solve $(1+x)dy - ydx = 0$.
(This is the differential form of a differential equation. Here, we assume that *y* denotes the dependent variable, and so $y' = \frac{dy}{dx}$. We simply divide the DE by *dx* to get $(1+x)\frac{dy}{dx} - y = 0$.)

As we noted earlier, when we subject a differential equation to certain initial conditions, we have an initial value problem. Formally, on some interval *I*, the problem

Solve: $\frac{d^n y}{dx^n} = F(x, y, y', \ldots, y^{(n-1)})$
Subject to: $y(x_0) = y_0$, $y'(x_0) = y_1$, …, $y^{(n-1)}(x_0) = y_{n-1}$

where y_0, y_1, …, y_{n-1} are constants, is called an **initial value problem (IVP)**.

Example 5: Solve the initial value problem $x^2 \frac{dy}{dx} = y - xy$, $y(-1) = -1$