Homogeneous Equations (Section 2.3)

If a function *f* possesses the property that

$$f(tx, ty) = t^{\alpha} f(x, y)$$

for some real number α , then *f* is said to be a **homogeneous function** of degree α .

Example 1: $f(x, y) = x^2 - 3xy + 5y^2$ is a homogeneous function of degree 2.

Example 2: $f(x, y) = x^3 + y^3$ is a homogeneous function of degree 3.

Example 3: $f(x, y) = \frac{x}{2y} + 4$ is a homogeneous function of degree 0.

Example 4: $f(x, y) = x^2 + y$ is not homogeneous. In general, the terms must have the same degree in order for the function to be homogeneous.

A first-order DE

$$M(x, y)dx + N(x, y)dy = 0$$

is written in *differential form*. It is said to be a **homogeneous differential equation** provided that *M* and *N* are homogeneous functions of the same degree, that is,

$$M(tx, ty) = t^{\alpha} M(x, y) \quad \text{and} \quad N(tx, ty) = t^{\alpha} N(x, y)$$

Substitutions can be used to solve homogeneous differential equations.

If *M* and *N* are homogeneous functions of degree α , then

$$M(x, y) = x^{\alpha} M(1, u) \text{ and } N(x, y) = x^{\alpha} N(1, u) \quad \text{where } u = \frac{y}{x}$$

and

$$M(x, y) = y^{\alpha} M(v, 1) \text{ and } N(x, y) = y^{\alpha} N(v, 1) \quad \text{where } v = \frac{x}{y}.$$

To understand this concept, consider $M(x, y) = x^3 + y^3$.

In solving homogeneous DEs, we can use the substitutions $y = ux$ or $x = vy$, where u and v are new dependent variables. One of these substitutions will reduce the homogeneous equations to a separable first-order DE.

Upon making the substitution $y = ux$, a homogeneous equation of the form

$$M(x, y)dx + N(x, y)dy = 0$$

can be written as

$$x^{\alpha} M(1,u)dx + x^{\alpha} N(1,u)dy = 0$$
$$M(1,u)dx + N(1,u)dy = 0$$

Because $y = ux$, it follows that $dy = udx + xdu$. Making this substitution into the above equation, we have

$$M(1,u)dx + N(1,u)[udx + xdu] = 0$$

This can be solved by separation of variables.

<u>Example 5:</u> Solve $(x + y)dx + xdy = 0$.

<u>Example 6:</u> Solve $\frac{dy}{dx} = \frac{y - x}{y + x}$.

<u>Example 7:</u> Solve $\left(y + x\cot\frac{y}{x}\right)dx - xdy = 0$.