Exact Equations (Section 2.4)

If $z = f(x, y)$ is a function with continuous partial derivatives in a region *R*, then its differential is

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy$$

If $f(x, y) = c$, then $\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0dx + 0dy = 0$, that is,

$$\text{if } f(x, y) = c, \text{ then } \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy = 0.$$

If given a one parameter family of curves $f(x, y) = c$, we can generate a first-order differential equation by computing the differentials. For example, if $x^2 - 5xy + y^3 = c$, then we can create the differential equation

$$(2x - 5y)dx + (-5x + 3y^2)dy = 0 \quad *$$

(Note that $\frac{\partial f}{\partial x} = 2x - 5y$ and $\frac{\partial f}{\partial y} = -5x + 3y^2$.)

We shall prefer to turn the problem around. That is, given a DE such as (*), can we recognize it as being equivalent to $D(x^2 - 5xy + y^3) = 0$?

A differential expression of the form $M(x, y)dx + N(x, y)dy$ is called an **exact differential** in a region *R* of the *xy*-plane if there exists a function $f(x, y)$ such that

$$M(x, y)dx + N(x, y)dy = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy,$$

that is, if $M(x, y)dx + N(x, y)dy$ corresponds to the differential of some function $f(x, y)$. The DE $M(x, y)dx + N(x, y)dy = 0$ is called an **exact differential equation** if the expression on the left side is an exact differential.

Theorem: The differential equation $M(x, y)dx + N(x, y)dy = 0$ is exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Method of Solution

Step 1: Given $M(x,y)dx + N(x,y)dy = 0$, determine whether or not $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

If equality holds, then the equation is exact. Thus, there exists a function $f(x,y)$ such that $\frac{\partial f}{\partial x} = M(x,y)$ and $\frac{\partial f}{\partial y} = N(x,y)$.

Step 2: Integrate both sides of $\frac{\partial f}{\partial x} = M(x,y)$ with respect to *x* (or integrate both sides of $\frac{\partial f}{\partial y} = N(x,y)$ with respect to *y*).

We can find *f* by integrating $M(x,y)$ with respect to *x*.

$$f(x,y) = \int M(x,y)dx + g(y)$$

In integrating with respect to *x*, we hold *y* constant. Consequently, the constant of integration could certainly involve *y*. Rather than representing the constant of integration by *c* as we usually do, we represent here by *g*(*y*).

Had we integrated $N(x,y)$ with respect to *y*, the constant of integration could have involved *x*. Thus we represent the constant of integration in this case by *h*(*x*).

Step 3: Determine *g*(*y*) or *h*(*x*) by taking the partial with respect to *y* (or *x*) and setting this result equal to $N(x,y)$ or $M(x,y)$.

To do this, we first differentiate our expression for $f(x,y)$ with respect to *y*. This gives

$$\frac{\partial}{\partial y}\left[\int M(x,y)dx + g(y)\right] = \frac{\partial}{\partial y}\int M(x,y)dx + g'(y).$$

But this expression is $N(x,y)$. (After all, $\frac{\partial f}{\partial y} = N(x,y)$.) Thus,

$$\frac{\partial}{\partial y}\int M(x,y)dx + g'(y) = N(x,y)$$

Solving for $g'(y)$, we obtain

$$g'(y) = N(x,y) - \frac{\partial}{\partial y}\int M(x,y)dx$$

Integrating both sides of this equation with respect to *y* allows us to determine *g*(*y*).

Step 4: Determine the solution.

We will have an implicit solution $f(x, y) = c$.

<u>Example:</u> Solve $2xydx + (x^2 - 1)dy = 0$.

Step 1: Determine whether or not $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

$\frac{\partial}{\partial y}(2xy) = 2x$, $\frac{\partial}{\partial y}(x^2 - 1) = 2x$. They are equal, and so the DE is exact. Thus, there exists a function $f(x, y)$ in which $\frac{\partial f}{\partial x} = 2xy$ and $\frac{\partial f}{\partial y} = x^2 - 1$. We must now find that function.

Step 2: Integrate both sides of $\frac{\partial f}{\partial x} = 2xy$ with respect to *x*.

$$\int \frac{\partial f}{\partial x} dx = \int 2xydx$$
$$f(x, y) = x^2 y + g(y)$$

Step 3: Determine g(*y*) by first taking the partial with respect to *y* and setting this result equal to $N(x, y)$.

$$\frac{\partial}{\partial y} f(x, y) = \frac{\partial}{\partial y}\left[x^2 y + g(y)\right]$$
$$\frac{\partial f}{\partial y} = x^2 + g'(y) = x^2 - 1$$
$$g'(y) = -1$$
$$g(y) = -y$$

Step 4: Determine the solution.

$$f(x, y) = x^2 y - y$$

Thus, the solution is $x^2 y - y = c$.

Note: This equation could have been solved by separation of variables.