Linear Equations - The Integrating Factor (Section 2.5)

First Order Linear ODEs

A first order DE of the form

$$a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$$

is a first order linear ordinary differential equation. An equation of this form is commonly referred to as a **linear equation**.

When $g(x) = 0$, the linear equation is said to be **homogeneous**.
When $g(x) \neq 0$, the linear equation is said to be **nonhomogeneous**.

When we divide both sides by the leading coefficient $a_1(x)$, we obtain a linear equation in **standard form**.

$$\frac{dy}{dx} + P(x)y = f(x)$$

We want to find a solution of the linear equation in standard form over an interval on which both *P* and *f* are continuous.

The Solution of a Linear Equation in Standard Form

The solution of the DE

$$\frac{dy}{dx} + P(x)y = f(x)$$

has the feature to it that it is the sum of two solutions, y_c and y_p, where y_c is the solution of the associated homogeneous equation

$$\frac{dy}{dx} + P(x)y = 0$$

and where y_p is a particular solution. We shall refer to y_c as the complementary function. To see that the sum of y_c and y_p is a solution to the DE, note the following. Let $y = y_c + y_p$. Then,

$$\frac{dy}{dx} + P(x)y = \frac{d}{dx}[y_c + y_p] + P(x)[y_c + y_p]$$

$$= \frac{dy_c}{dx} + \frac{dy_p}{dx} + P(x)y_c + P(x)y_p$$
$$= \underbrace{\left[\frac{dy_c}{dx} + P(x)y_c\right]}_{=0} + \underbrace{\left[\frac{dy_p}{dx} + P(x)y_p\right]}_{=f(x)}$$
$$= f(x)$$

Because the homogeneous equation

$$\frac{dy_c}{dx} + P(x)y_c = 0$$

is separable, we can readily solve for y_c.

<u>The Homogeneous Solution</u>

We can find the homogeneous solution by separating variables.

$$\frac{dy_c}{dx} + P(x)y_c = 0$$
$$\frac{dy_c}{dx} = -P(x)y_c$$
$$\frac{dy_c}{y_c} = -P(x)dx$$
$$\int \frac{dy_c}{y_c} = -\int P(x)dx$$
$$\ln|y_c| = -\int P(x)dx + c$$
$$|y_c| = e^{-\int P(x)dx + c}$$
$$|y_c| = e^c e^{-\int P(x)dx}$$
$$y_c = ce^{-\int P(x)dx}$$

<u>The Particular Solution</u>

We find the particular solution by a method called **variation of parameters**. The idea is that the particular solution will look like the homogeneous solution

$$y_c = ce^{-\int P(x)dx}.$$

except that *c* is replaced by a function *u*.

$$y_p = u(x)e^{-\int P(x)dx}.$$

Assuming that this is a particular solution, we plug $y_p = u(x)e^{-\int P(x)dx}$ back into the original DE and solve for *u*.

$$\begin{aligned}
&\frac{d}{dx}\left[u(x)e^{-\int P(x)dx}\right] + P(x)u(x)e^{-\int P(x)dx} = f(x) \\
&\frac{du}{dx}e^{-\int P(x)dx} + u(x)\frac{d}{dx}\left[e^{-\int P(x)dx}\right] + P(x)u(x)e^{-\int P(x)dx} = f(x) \\
&\frac{du}{dx}e^{-\int P(x)dx} - P(x)u(x)e^{-\int P(x)dx} + P(x)u(x)e^{-\int P(x)dx} = f(x) \\
&\frac{du}{dx}e^{-\int P(x)dx} = f(x)
\end{aligned}$$

We can now separate the variables and integrate.

$$\begin{aligned}
du &= \frac{f(x)}{e^{-\int P(x)dx}}dx \\
du &= f(x)\cdot e^{\int P(x)dx}dx \\
\int du &= \int f(x)\cdot e^{\int P(x)dx}dx \\
u &= \int f(x)\cdot e^{\int P(x)dx}dx
\end{aligned}$$

Replacing *u*(*x*) by $\int f(x)\cdot e^{\int P(x)dx}dx$ in $y_p = u(x)e^{-\int P(x)dx}$, we have

$$y_p = e^{-\int P(x)dx}\int f(x)e^{\int P(x)dx}dx$$

Thus,

$$y = \underbrace{ce^{-\int P(x)dx}}_{y_c} + \underbrace{e^{-\int P(x)dx}\int f(x)e^{\int P(x)dx}dx}_{y_p}$$

And so, if

$$\frac{dy}{dx} + P(x)y = f(x)$$

has a solution, it must be of the form

$$y = \underbrace{ce^{-\int P(x)dx}}_{y_c} + \underbrace{e^{-\int P(x)dx}\int f(x)e^{\int P(x)dx}dx}_{y_p} \quad *$$

While we could memorize *, we won't. Rather, we'll note the following.

(1) If we multiply * by $e^{\int P(x)dx}$, we get

$$ye^{\int P(x)dx} = c + \int f(x)e^{\int P(x)dx}dx.$$

(2) If we then differentiate both sides of the above equation, we obtain

$$\frac{d}{dx}\left[ye^{\int P(x)dx}\right] = f(x)e^{\int P(x)dx}$$
$$e^{\int P(x)dx}\frac{dy}{dx} + P(x)e^{\int P(x)dx}y = f(x)e^{\int P(x)dx}$$

(3) If we now divide the last result by $e^{\int P(x)dx}$, we obtain the original differential equation, $\frac{dy}{dx} + P(x)y = f(x)$.

The three manipulations above suggest a method of solution. If we start with the original equation and reverse the above three steps, we can arrive at the solution.

Method of Solution

(1) Find $e^{\int P(x)dx}$. This is called an **integrating factor**. Multiply $\frac{dy}{dx} + P(x)y = f(x)$ by the integrating factor. The left side of the equation is automatically the derivative of the product of the integrating factor and *y*.
(2) Integrate both sides of the equation from s step 1.
(3) Solve for *y*. Often, this requires that we divide through by the integrating factor.