Picard's Method (Section 2.8)

An **integral equation** is an equation of the form

$$y(x) = c + \int_{x_0}^{x} f(t, y(t))dt$$

An initial value problem can be expressed as an integral equation. To see this, consider the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Integrating both sides of the equation with respect to *x*, we obtain the following.

$$\int_{x_0}^{x} \frac{dy}{dt} dt = \int_{x_0}^{x} f(t, y(t))dt$$
$$y(x) - y(x_0) = \int_{x_0}^{x} f(t, y(t))dt$$
$$y(x) = y(x_0) + \int_{x_0}^{x} f(t, y(t))dt$$

Since $y(x_0) = y_0$, we have the following.

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))dt \quad (*)$$

Now, suppose that $y_0(x)$ is an approximation to the solution of the above integral equation. Then, $f(x, y_0(x))$ is a known function depending on *x* and can be integrated. When we replace $y(t)$ by in (*), we obtain a new function.

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0(t))dt$$

Repeating the procedure, we obtain the following.

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))dt$$

Continuing in this we, we obtain a sequence of functions whose n^{th} term is

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))dt, \ n = 1, 2, 3, \ldots .$$

Repetitive use of the above formula is known as Picard's method of approximation.

Each of the y_n 's are called *Picard iterates*.

A very reasonable question is this: why should these y_n approximate the solution of the IVP? The answer is somewhat unsatisfactory at this point. The answer really lies in the following theorem.

Theorem: If $\{y_n\}$ is a sequence of continuous functions on some interval *I* containing x_0 in its interior, and if $\{y_n\}$ converges uniformly* on *I*, then

$$y(x) = \lim_{n\to\infty} y_n(x) = y_0 + \lim_{n\to\infty}\int_{x_0}^{x} f(t, y_{n-1}(t))dt = y_0 + \int_{x_0}^{x} f(t, y(t))dt$$

on *I* and $y(x_0) = y_0$.

In view of another theorem, this then means that the function *y* will be a solution of the IVP.

<u>Example:</u> Use Picard's method to find the solution of the IVP $y' = y - 1, y(0) = 2$.

*Let $\{y_n\}$ be a sequence of real-valued functions defined on a set *I*. We that that the sequence of functions converges uniformly to the function *y* on *I*, if given $\varepsilon > 0$, there exists $N(\varepsilon) \in \bullet$ such that for all $x \in I$, $|y_n(x) - y(x)| < \varepsilon$ whenever $n \geq N$.