MTH 202: Exam 1 Name: key
Time: 30 minutes

Use of notes, note cards, cellphones or PDAs are not allowed during the quiz.

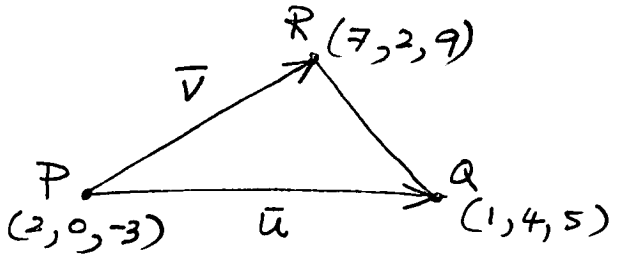
(1) Find the directional cosines of the vector $< 4, 2, 1 >$.
Show all the reasoning, in a step by step manner, which leads to the logical conclusion.

Let $\bar{v} = \langle 4, 2, 1 \rangle$.

$\Rightarrow \|\bar{v}\| = \sqrt{16+4+1} = \sqrt{21}$.

$\therefore \cos\alpha = \frac{4}{\sqrt{21}}, \cos\beta = \frac{2}{\sqrt{21}}, \cos\gamma = \frac{1}{\sqrt{21}}$ //

(2) Find the area of the triangle with vertices $(2, 0, -3)$, $(1, 4, 5)$ and $(7, 2, 9)$ using vectors. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.



Let $\overrightarrow{PQ} = \bar{u}$ and $\overrightarrow{PR} = \bar{v}$ as seen in the figure.

$\Rightarrow \bar{u} = \langle -1, 4, 8 \rangle$ and

$\bar{v} = \langle 5, 2, 12 \rangle$.

$\Rightarrow \bar{u} \times \bar{v} = \langle 32, 52, -22 \rangle = 2\langle 16, 26, -11 \rangle$

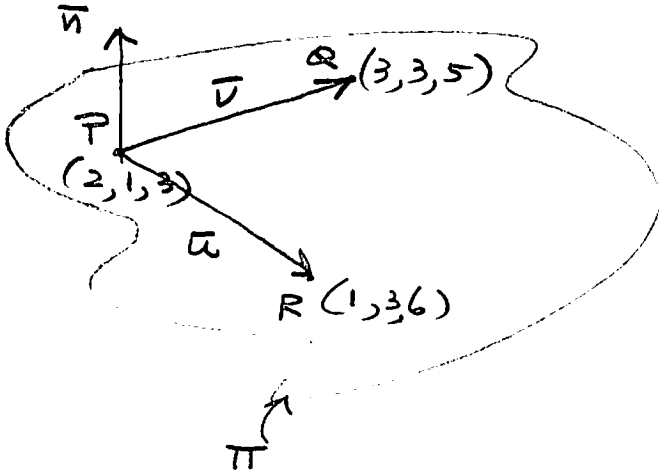
$\Rightarrow \|\bar{u} \times \bar{v}\| = 2\sqrt{256 + 676 + 121}$

$= 2\sqrt{1053}$

$= 18\sqrt{13}$

$\therefore$ The area of the triangle is $9\sqrt{13}$ s.u. //

(3) Find the equation of the plane containing the points $(2, 1, 3)$, $(3, 3, 5)$, and $(1, 3, 6)$. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.



Let π be the plane containing the points P, Q & R. (See figure.)
Let $\bar{u} = \vec{PR}$ and $\bar{v} = \vec{PQ}$.

Then $\bar{u} = \langle -1, 2, 3 \rangle$ and
$\bar{v} = \langle 1, 2, 2 \rangle$.

$\Rightarrow \bar{n} = \bar{u} \times \bar{v} = \langle -2, 5, -4 \rangle$

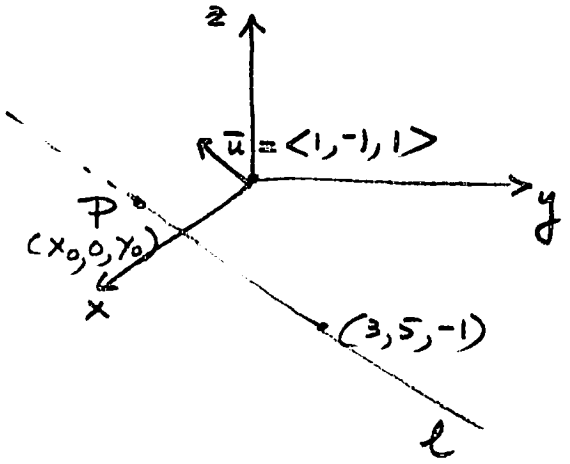
$\therefore$ The equation of the plane is:

$$-2(x-2) + 5(y-1) - 4(z-3) = 0$$

or

$$2x - 5y + 4z - 11 = 0 \;//$$

(4) Find the point where the line which passes through the point $(3, 5, -1)$ and is parallel to $< 1, -1, 1 >$ pierces the $xz-$plane. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.



Let $\bar{u} = \langle 1, -1, 1 \rangle$, and ℓ be the line passing thro' $(3, 5, -1)$ and parallel to $\bar{u}$. Let $P(x_0, 0, y_0)$ be the point where ℓ pierces the xz-plane.

The vector equation of ℓ is:

$\langle x, y, z \rangle = \langle 3, 5, -1 \rangle + t \langle 1, -1, 1 \rangle$ for some parameter t.

Let t_0 be the value of t corresponds to P.

Then

$\langle x_0, 0, z_0 \rangle = \langle 3, 5, -1 \rangle + t_0 \langle 1, -1, 1 \rangle$

$\Rightarrow \quad 0 = 5 - t_0 \quad$ or $\quad t_0 = 5.$

$\therefore \; x_0 = 3 + 5 = 8 \quad$ and $\quad z_0 = -1 + 5 = 4.$

ie, $P = (8, 0, 4)$. //

(5) Find the sine of the angle between $\overline{u} =< 1, 1, 1 >$ and $\overline{v} =< 2, -1, 3 >$ Most simplified exact numerical answer is requried. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.

$\bar{u} = \langle 1, 1, 1 \rangle \Longrightarrow \|\bar{u}\| = \sqrt{3}.$

$\bar{v} = \langle 2, -1, 3 \rangle \Longrightarrow \|\bar{v}\| = \sqrt{14}.$

$\bar{u} \times \bar{v} = \langle 4, -1, -3 \rangle$

$\Rightarrow \|\bar{u} \times \bar{v}\| = \sqrt{26}$

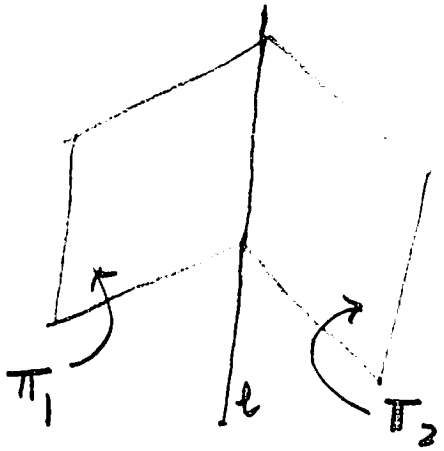
Since, $\|\bar{u} \times \bar{v}\| = \|\bar{u}\| \|\bar{v}\| \sin\theta$, where θ is the angle between $\bar{u}$ & $\bar{v}$,

$\Rightarrow \sin\theta = \dfrac{\|\bar{u} \times \bar{v}\|}{\|\bar{u}\| \|\bar{v}\|}.$

$\Rightarrow \sin\theta = \dfrac{\sqrt{26}}{\sqrt{3}\sqrt{14}}$

$\Rightarrow \sin\theta = \sqrt{\dfrac{13}{21}}.$ //

(6) Find the vector equation of the line of intersection of the planes $x + y + z - 4 = 0$ and $2x - y + z - 2 = 0$. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.



Let π_1 be the plane $x+y+z-4=0$ and π_2 be the plane $2x-y+z-2=0$.

Let ℓ be the line of intersection of π_1 and π_2.

$\bar{n}_1 = \langle 1,1,1 \rangle \perp \pi_1$ and therefore, $\perp \ell$, since ℓ is on π_1.

$\bar{n}_2 = \langle 2,-1,1 \rangle \perp \pi_2$ and therefore, $\perp \ell$, since ℓ is on π_2.

$\Rightarrow \bar{u} = \bar{n}_1 \times \bar{n}_2 = \langle 2,1,-3 \rangle \parallel \ell$, since $\bar{u}$ is $\perp$ to both $\bar{n}_1$ & $\bar{n}_2$ as ℓ does.

Assume ℓ intersects the xy-plane at $P(x_0, y_0, 0)$.

P is on $\pi_1 \Rightarrow x_0 + y_0 = 4$. — ①

P is on $\pi_2 \Rightarrow 2x_0 - y_0 = 2$. — ②

① + ② $\Rightarrow 3x_0 = 6$ or $x_0 = 2$. Therefore, $y_0 = 2$.

∴ The vector equation of ℓ is:

$\langle x, y, z \rangle = \langle 2,2,0 \rangle + t\langle 2,1,-3 \rangle$ for some parameter t. //

(7) Find the point of intersection of the line $x = 3 - t$, $y = 5 + 3t$, $z = -1 - 4t$ and the line $x = 8 + 2t$, $y = -6 - 4t$, $z = 5 + t$, if they intersect. If not, then decide if they are parallel or skew. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.

Suppose the point of intersection is (x_0, y_0, z_0).

Then for some t_1,

$x_0 = 3 - t_1$ —①
$y_0 = 5 + 3t_1$ —②
$z_0 = -1 - 4t_1$ —③

and

for some t_2,

$x_0 = 8 + 2t_2$ —④
$y_0 = -6 - 4t_2$ —⑤
$z_0 = 5 + t_2$ —⑥

①&④ $\Rightarrow t_1 + 2t_2 = -5$ —⑦

③&⑥ $\Rightarrow 4t_1 + t_2 = -6$ —⑧

⑦+(-2)⑧ $\Rightarrow -7t_1 = 7 \Rightarrow t_1 = -1$.

⑦ $\Rightarrow 2t_2 = -4 \Rightarrow t_2 = -2$.

Sub. $t_1 = -1$ in ①, ② & ③ $\Rightarrow$

$x_0 = 4$, $y_0 = 2$, $z_0 = 3$.

We have not used ⑥ yet. Use ⑥ to check.

Is $3 = 5 - 2$? The answer is yes.

∴ The point of intersection is $(4, 2, 3)$. //

(8) Name and sketch the quadric surface $x^2 - y^2 + z^2 + 2y = 1$. Make your sketch more accurate by identifying the necessary points on the surface. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.

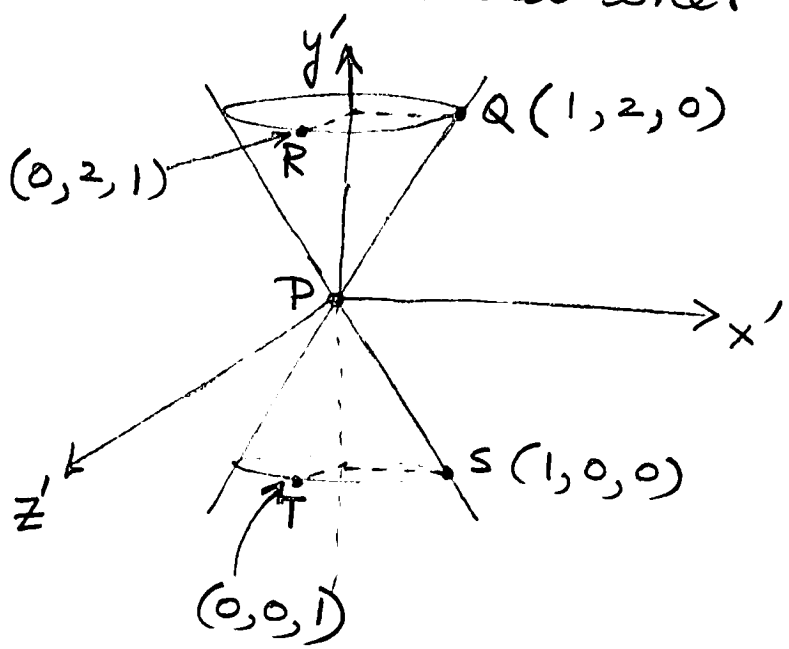
$$x^2 - y^2 + 2y + z^2 = 1$$

$$\Rightarrow x^2 - (y^2 - 2y + 1) + z^2 = 0$$

$$\Rightarrow x^2 - (y-1)^2 + z^2 = 0$$

$$\Rightarrow (y-1)^2 = x^2 + z^2.$$

This is a circular cone.



In xyz coord system:

$P \equiv (0,1,0)$.

(9) Transform the equation $z = \frac{1}{4}(x^2 + y^2)$ from rectangular coordinates to spherical coordinates. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.

In spherical coordinates:

$z = \rho\cos\phi$,

$x = \rho\sin\phi\cos\theta$, and

$y = \rho\sin\phi\sin\theta$.

$\therefore \quad \rho\cos\phi = \frac{1}{4}\left(\rho^2\sin^2\phi\cos^2\theta + \rho^2\sin^2\phi\sin^2\theta\right)$

$\Longrightarrow \quad 4\rho\cos\phi = \rho^2\sin^2\phi\left(\cos^2\theta + \sin^2\theta\right)$

$\Longrightarrow \quad 4\rho\cos\phi - \rho^2\sin^2\phi = 0$

$\Longrightarrow \quad \rho\left(4\cos\phi - \rho\sin^2\phi\right) = 0$

$\Longrightarrow \quad \rho = 0 \quad$ or $\quad 4\cos\phi = \rho\sin^2\phi$

$\rho = 0$ ✗

because the equation represents a circular paraboloid

$\therefore$ Equation is $4\cos\phi = \rho\sin^2\phi$

or $\quad \rho = 4\cot\phi\csc\phi \;;\; \phi \neq 0$ //

(10) Sketch the graph of the vector valued function $\bar{r}(t) = < 2 + \sin t, 3 - \cos t >$, $0 \leq t < 2\pi$ accurately. Describe the graph in words precisely. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.

Notice that $x(t) = 2 + \sin t$ and

$y(t) = 3 - \cos t$.

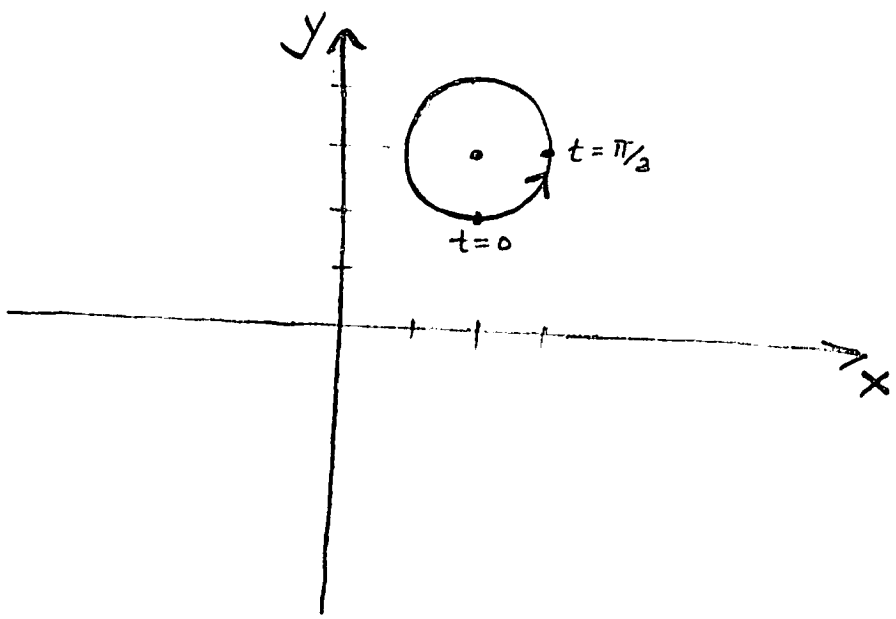
$\Rightarrow$ $x - 2 = \sin t$ and $y - 3 = -\cos t$.

$\Rightarrow$ $(x-2)^2 + (y-3)^2 = 1$.

This is a circle with center $(2,3)$ and radius 1,

When $t = 0$, $\bar{r}(0) = \langle 2, 2 \rangle$ and

When $t = \frac{\pi}{2}$, $\bar{r}\left(\frac{\pi}{2}\right) = \langle 3, 3 \rangle$.



∴ The graph is a circle with center $(2,3)$, radius 1 and orientation counterclockwise. //