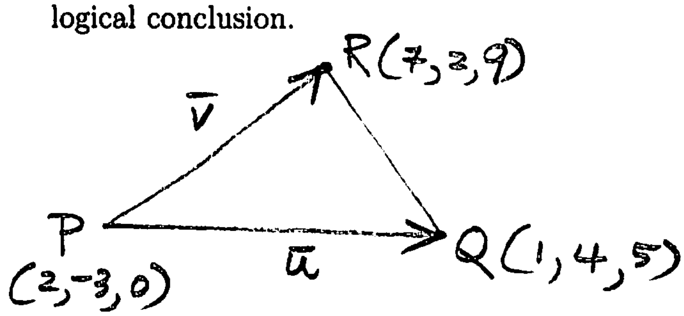
MTH 202: Exam 1 **Name:** key

Time: 30 minutes

Use of notes, note cards, cellphones or PDAs are not allowed during the quiz.

(1) Find the area of the triangle with vertices $(2,-3,0)$, $(1,4,5)$ and $(7,2,9)$ using vectors. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.



Let $\bar{u} = \overrightarrow{PQ}$ and $\bar{v} = \overrightarrow{PR}$ as shown.

$\Longrightarrow \bar{u} = \langle -1, 7, 5 \rangle$ and

$\bar{v} = \langle 5, 5, 9 \rangle$.

$\Longrightarrow \bar{u} \times \bar{v} = \langle 38, -34, -40 \rangle = 2\langle 19, -17, -20 \rangle$

$\Longrightarrow \|\bar{u} \times \bar{v}\| = 2\sqrt{1050}$

$= 10\sqrt{42}$

$\therefore$ The area of the triangle is $5\sqrt{42}$ s.u. //

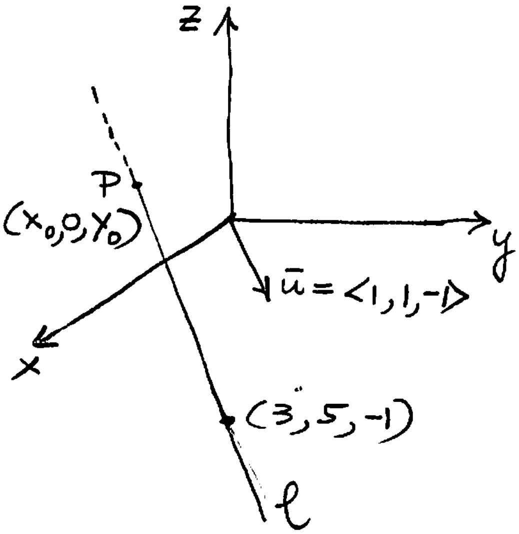
(2) Find the directional cosines of the vector $< 2, 4, 1 >$.
Show all the reasoning, in a step by step manner, which leads to the logical conclusion.

Let $\bar{v} = <2, 4, 1>$.

$\Longrightarrow \|\bar{v}\| = \sqrt{4+16+1} = \sqrt{21}$.

$\therefore \cos\alpha = \frac{2}{\sqrt{21}}, \quad \cos\beta = \frac{4}{\sqrt{21}}, \quad \cos\gamma = \frac{1}{\sqrt{21}}$. //

(3) Find the point where the line which passes through the point $(3, 5, -1)$ and is parallel to $< 1, 1, -1 >$ pierces the $xz-$plane. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.



Let $\bar{u} = <1,1,-1>$, and ℓ be the line passing thro' $(3,5,-1)$ and Parallel to $\bar{u}$.

Let $P(x_0, 0, Y_0)$ be the point where ℓ pierces the xz-plane.

The vector equation of ℓ is:

$$<x, y, z> = <3,5,-1> + t<1,1,-1> \text{ for some parameter } t.$$

Let t_0 be the value of t corresponds to P.

Then

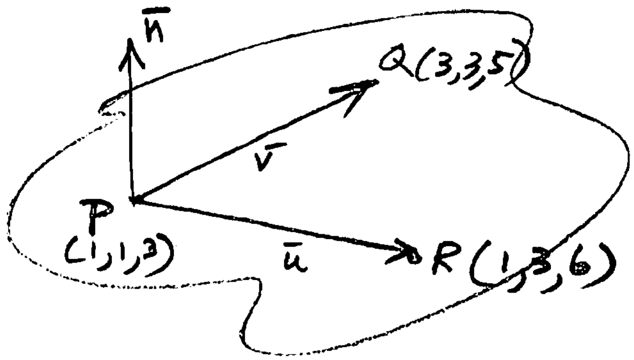
$$<x_0, 0, z_0> = <3,5,-1> + t_0<1,1,-1>$$

$$\Rightarrow \quad 0 = 5 + t_0 \quad \text{or} \quad t_0 = -5.$$

$$\therefore \quad x_0 = 3-5 = -2 \quad \text{and} \quad z_0 = -1+5 = 4.$$

ie, $P \equiv (-2, 0, 4)$. //

(4) Find the equation of the plane containing the points $(1,1,3)$, $(3,3,5)$, and $(1,3,6)$. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.



Let Π be the plane containing the points P, Q & R. (See figure.) Let $\bar{u} = \overrightarrow{PR}$ and $\bar{v} = \overrightarrow{PQ}$.

Then $\bar{u} = \langle 0, 2, 3 \rangle$ and

$\bar{v} = \langle 2, 2, 2 \rangle$.

$\Longrightarrow \quad \bar{n} = \bar{u} \times \bar{v} = \langle -2, 6, -4 \rangle$

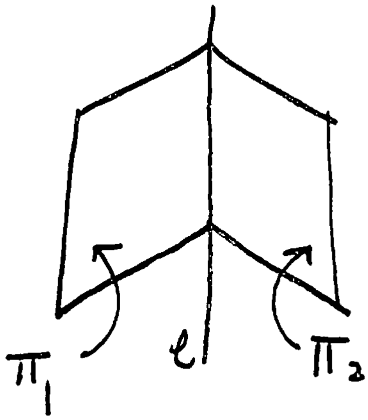
$\therefore$ The equation of the plane is:

$$-\overset{1}{\cancel{2}}(x-1) + \overset{3}{\cancel{6}}(y-1) - \overset{2}{\cancel{4}}(z-3) = 0$$

or

$$x - 3y + 2z - 4 = 0 \;//$$

(5) Find the vector equation of the line of intersection of the planes $x + y + z - 4 = 0$ and $2x - y + z - 2 = 0$. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.



Let π_1 be the plane $x+y+z-4=0$ and π_2 be the plane $2x-y+z-2=0$. Let ℓ be the line of intersection of π_1 and π_2.

$\bar{n}_1 = \langle 1,1,1 \rangle \perp \pi_1$ and therefore, $\perp \ell$, since ℓ is on π_1.

$\bar{n}_2 = \langle 2,-1,1 \rangle \perp \pi_2$ and therefore, $\perp \ell$, since ℓ is on π_2.

$\Rightarrow \bar{u} = \bar{n}_1 \times \bar{n}_2 = \langle 2,1,-3 \rangle \parallel \ell$, since $\bar{u}$ is $\perp$ to both $\bar{n}_1$ & $\bar{n}_2$ as ℓ does.

Assume ℓ intersects the xy-plane at $P(x_0, y_0, 0)$.

P is on $\pi_1 \Longrightarrow x_0 + y_0 = 4$ — ①.

P is on $\pi_2 \Longrightarrow 2x_0 - y_0 = 2$ — ②.

① + ② $\Longrightarrow 3x_0 = 6$ or $x_0 = 2$. Therefore, $y_0 = 2$.

$\therefore$ The vector equation of ℓ is:

$\langle x,y,z \rangle = \langle 2,2,0 \rangle + t\langle 2,1,-3 \rangle$ for some parameter t.

(6) Find the sine of the angle between $\overline{u} =< 2, 1, 1 >$ and $\overline{v} =< 1, -1, 3 >$ Most simplified exact numerical answer is requried. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.

$$\bar{u} = <2, 1, 1> \implies \|\bar{u}\| = \sqrt{6}.$$

$$\bar{v} = <1, -1, 3> \implies \|\bar{v}\| = \sqrt{11}.$$

$$\bar{u} \times \bar{v} = <4, -5, -3>$$

$$\implies \|\bar{u} \times \bar{v}\| = \sqrt{50}$$

Since $\|\bar{u} \times \bar{v}\| = \|\bar{u}\| \|\bar{v}\| \sin\theta$, where θ is the angle between $\bar{u}$ & v,

$$\implies \sin\theta = \frac{\|\bar{u} \times \bar{v}\|}{\|\bar{u}\| \|\bar{v}\|}.$$

$$\implies \sin\theta = \frac{\sqrt{50}}{\sqrt{6}\sqrt{11}}$$

$$\implies \sin\theta = \frac{5}{\sqrt{33}}.$$

(7) Transform the equation $z = \frac{1}{4}(x^2 + y^2)$ from rectangular coordinates to spherical coordinates. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.

In spherical coordinates:

$$z = \rho \cos\phi$$

$$x = \rho \sin\phi \cos\theta, \text{ and}$$

$$y = \rho \sin\phi \sin\theta.$$

$$\therefore \rho \cos\phi = \frac{1}{4}\left(\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta\right)$$

$$\Longrightarrow 4\rho \cos\phi = \rho^2 \sin^2\phi \left(\cos^2\theta + \sin^2\theta\right)$$

$$\Longrightarrow 4\rho \cos\phi - \rho^2 \sin^2\phi = 0$$

$$\Longrightarrow \rho\left(4\cos\phi - \rho \sin^2\phi\right) = 0$$

$$\Longrightarrow \rho = 0 \quad \text{or} \quad 4\cos\phi = \rho \sin^2\phi$$

X
because the equation represents a circular paraboloid.

$\therefore$ The equation is $4\cos\phi = \rho \sin^2\phi$

or $\rho = 4\cot\phi \csc\phi \; ; \; \phi \neq 0.$ //

(8) Find the point of intersection of the line $x = 3 - t$, $y = 5 + 3t$, $z = -1 - 4t$ and the line $x = 8 + 2t$, $y = -6 - 4t$, $z = 5 + t$, if they intersect. If not, then decide if they are parallel or skew. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.

Suppose the point of intersection is (x_0, y_0, z_0).

Then for some t_1,

$x_0 = 3 - t_1$ — ①

$y_0 = 5 + 3t_1$ — ②

$z_0 = -1 - 4t_1$ — ③

and for some t_2,

$x_0 = 8 + 2t_2$ — ④

$y_0 = -6 - 4t_2$ — ⑤

$z_0 = 5 + t_2$ — ⑥

① & ④ $\Longrightarrow$ $t_1 + 2t_2 = -5$ — ⑦

③ & ⑥ $\Longrightarrow$ $4t_1 + t_2 = -6$ — ⑧

⑦ + (-2)·⑧ $\Longrightarrow$ $-7t_1 = 7 \Longrightarrow t_1 = -1$.

⑦ $\Longrightarrow$ $2t_2 = -4 \Longrightarrow t_2 = -2$.

Sub. $t_1 = -1$ in ①, ② & ③ $\Longrightarrow$

$x_0 = 4$, $y_0 = 2$, $z_0 = 3$.

We have not used ⑥ yet. Use ⑥ to check.

Is $3 = 5 - 2$? The answer is yes.

$\therefore$ The point of intersection is $(4, 2, 3)$.

(9) Sketch the graph of the vector valued function $\overline{r}(t) = < 2 - \sin t, 3 + \cos t >$, $0 \leq t < 2\pi$ accurately. Describe the graph in words precisely. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.

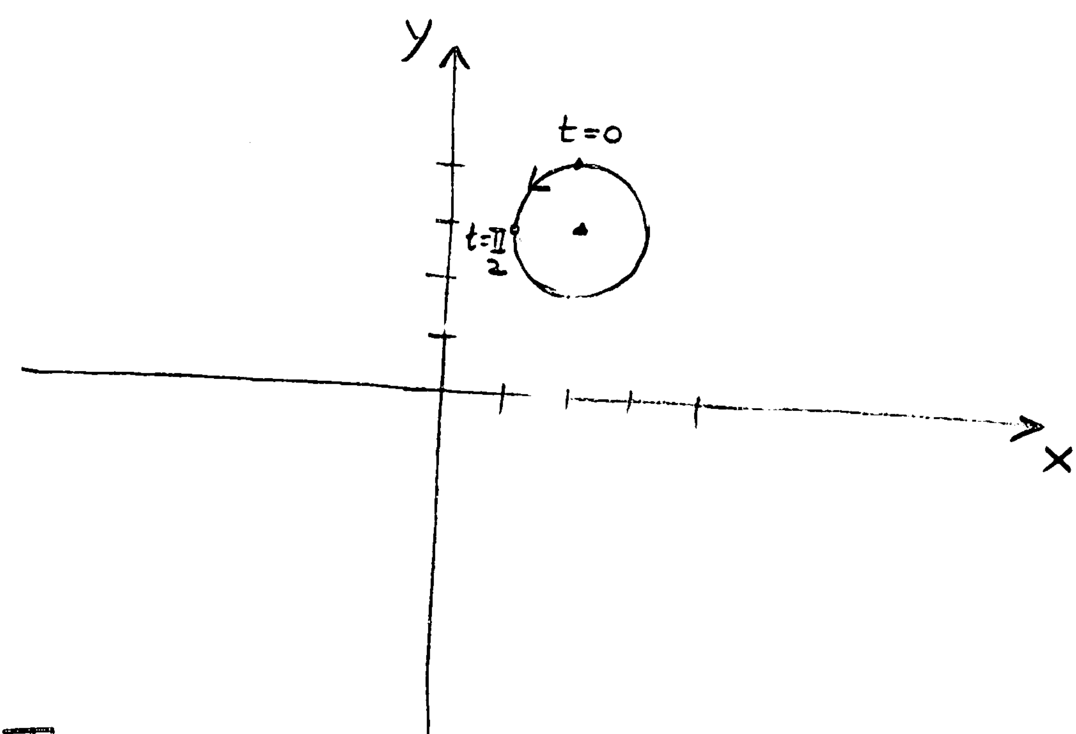
Notice that $x(t) = 2 - \sin t$ and
$y(t) = 3 + \cos t$.

$\Longrightarrow$ $x - 2 = -\sin t$ and $y - 3 = \cos t$

$\Longrightarrow$ $(x-2)^2 + (y-3)^2 = 1$.

This is a circle with center $(2, 3)$ and radius 1.

When $t = 0$, $\bar{r}(0) = \langle 2, 4 \rangle$ and
when $t = \frac{\pi}{2}$, $\bar{r}(\frac{\pi}{2}) = \langle 1, 3 \rangle$.



The graph is a circle with center $(2, 3)$, radius 1 and orientation counterclockwise.

(10) Name and sketch the quadric surface $x^2 - y^2 + z^2 + 2y = 1$. Make your sketch more accurate by identifying the necessary points on the surface. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.

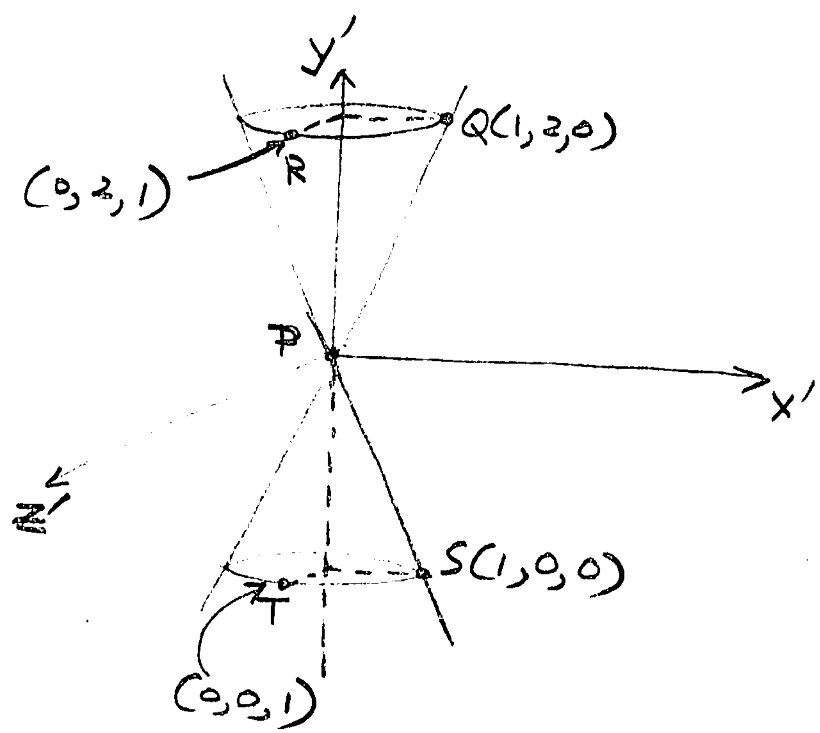
$$x^2 - y^2 + 2y + z^2 = 1$$

$$\Longrightarrow x^2 - (y^2 - 2y + 1) + z^2 = 0$$

$$\Longrightarrow x^2 - (y-1)^2 + z^2 = 0$$

$$\Longrightarrow (y-1)^2 = x^2 + y^2.$$

This is a circular cone.



In xyz coord. system; $P \equiv (0,1,0)$.