MTH 202: Exam 2 **Name:** key
Time: 30 minutes

Use of notes, note cards, cellphones or PDAs are not allowed during the quiz.

(1) Find the arc length of the curve given by $\bar{r}(t) = < \cos t, \sin t, \sqrt{t^3} >$ for $0 \le t \le \frac{20}{3}$.
Show all your reasoning, in a step by step manner, which leads to the logical conclusion.

$$\bar{r}'(t) = < -\sin t, \cos t, \tfrac{3}{2} t^{1/2} >$$

$$\|\bar{r}'(t)\| = \sqrt{\sin^2 t + \cos^2 t + \tfrac{9}{4} t} = \sqrt{1 + \tfrac{9}{4} t}$$

$$\Longrightarrow L = \int_0^{\frac{20}{3}} \sqrt{1 + \tfrac{9}{4} t}\, dt$$

$$\int \sqrt{1 + \tfrac{9}{4} t}\, dt$$

Let $u = 1 + \frac{9}{4} t$,
$\longrightarrow du = \frac{9}{4} dt$

$$= \frac{4}{9} \int u^{1/2} du$$

$$= \frac{4}{9} \cdot \frac{2}{3} u^{3/2} + C$$

$$= \frac{8}{27} \left(1 + \tfrac{9}{4} t\right)^{3/2} + C$$

$$= \frac{8}{27} \left(1 + \tfrac{9}{4} t\right)^{3/2} \Big|_0^{\frac{20}{3}}$$

$$= \frac{8}{27} \left[16^{3/2} - 1^{3/2} \right]$$

$$= \frac{8}{27} \left[64 - 1 \right]$$

$$= \frac{56}{3} \text{ length units}$$

(2) Find the unit tangent vector and the unit normal vector to $\overline{r}(t) = <\ln t, t>$ at $t = 2$.

Show all the reasoning, in a step by step manner, which leads to the logical conclusion.

$$\bar{r}'(t) = \left\langle \frac{1}{t}, 1 \right\rangle \Longrightarrow \|\bar{r}'(t)\| = \sqrt{\frac{1}{t^2} + 1} = \frac{\sqrt{1+t^2}}{t}$$

(We are in the neighbourhood of 2.)

$$\Longrightarrow \bar{T}(t) = \frac{\langle 1, t \rangle}{\sqrt{1+t^2}} = (1+t^2)^{-\frac{1}{2}} \langle 1, t \rangle$$

$$\Longrightarrow \bar{T}'(t) = -\frac{\cancel{2}t}{\cancel{2}(1+t^2)^{3/2}} \langle 1, t \rangle + \frac{1}{(1+t^2)^{1/2}} \langle 0, 1 \rangle$$

$$\Longrightarrow \bar{T}'(2) = \frac{-2}{5\sqrt{5}} \langle 1, 2 \rangle + \frac{1}{\sqrt{5}} \langle 0, 1 \rangle = \frac{1}{5\sqrt{5}} \Big[\langle -2, -4 \rangle + \langle 0, 5 \rangle \Big] = \frac{1}{5\sqrt{5}} \langle -2, 1 \rangle$$

$$\Longrightarrow \|\bar{T}'(2)\| = \frac{1}{5\cancel{\sqrt{5}}} \cdot \cancel{\sqrt{5}} = \frac{1}{5},$$

$$\therefore \bar{N}(2) = \frac{1}{\sqrt{5}} \langle -2, 1 \rangle \;//$$

$$\bar{T}(2) = \frac{1}{\sqrt{5}} \langle 1, 2 \rangle \;//$$

(3)Find the curvature for $\overline{r}(t) = < e^3, \sqrt{2}t, e^{-t} >$ at $t = 0$.
Show all your reasoning, in a step by step manner, which leads to the logical conclusion.

$$\bar{r}'(t) = <0, \sqrt{2}, -e^{-t}> \Longrightarrow \bar{r}'(0) = <0, \sqrt{2}, -1>$$

$$\bar{r}''(t) = <0, 0, e^{-t}> \Longrightarrow r''(0) = <0, 0, 1>$$

$$\Longrightarrow \bar{r}' \times \bar{r}''(0) = <\sqrt{2}, 0, 0>$$

$$\Longrightarrow \|\bar{r}' \times \bar{r}''(0)\| = \sqrt{2}.$$

$$\bar{r}'(0) = <0, \sqrt{2}, -1> \Longrightarrow \|\bar{r}'(0)\| = \sqrt{3}$$

$$\therefore \kappa(0) = \underline{\underline{\frac{\sqrt{2}}{3\sqrt{3}}}}$$

(4) Suppose $\bar{v}(t_0) =< 4, 0, -3 >$ and $\bar{a}(t_0) =< 1, -1, 2 >$ are the velocity and the acceleration of a particle at some instant $t = t_0$. Find a_T, a_N, $\bar{T}$, and $\bar{N}$ at that instant.

Show all your reasoning, in a step by step manner, which leads to the logical conclusion.

$$\|\bar{v}(t_0)\| = 5.$$

$$\Longrightarrow \bar{T}(t_0) = \left\langle \frac{4}{5}, 0, \frac{-3}{5} \right\rangle //$$

$$\bar{v}\cdot\bar{a}(t_0) = 4+0-6 = -2.$$

$$\bar{v}\times\bar{a}(0) = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ 4 & 0 & -3 \\ 1 & -1 & 2 \end{vmatrix}$$

$$= \langle -3, -11, -4 \rangle$$

$$\Rightarrow \|\bar{v}\times\bar{a}(0)\| = \sqrt{146}.$$

$$\therefore a_T = \frac{-2}{5} // , \quad a_N = \frac{\sqrt{146}}{5} //$$

$$\bar{a}(t_0) = a_T \bar{T}(t_0) + a_N \bar{N}(t_0).$$

$$\Longrightarrow \langle 1, -1, 2 \rangle = \frac{-2}{5}\left\langle \frac{4}{5}, 0, \frac{-3}{5} \right\rangle + \frac{\sqrt{146}}{5}\bar{N}(t_0)$$

$$\Longrightarrow \frac{\sqrt{146}}{5}\bar{N}(t_0) = \left\langle \frac{33}{25}, -1, \frac{44}{25} \right\rangle$$

$$\Longrightarrow \bar{N}(t_0) = \frac{1}{\sqrt{146}}\left\langle \frac{33}{5}, -5, \frac{44}{5} \right\rangle //$$

(5) A shell is fired from ground level at an elevation of $\frac{\pi}{3}$ and strikes a target 6000 meters away.

(a) Obtain the vector equation of the path of the shell.

(b) Calculate the muzzle speed of the shell.

The gravitational constant $g \approx 9.8\mathbf{m/sec}^2$.

Show all your reasoning, in a step by step manner, which leads to the logical conclusion.

(a) $\bar{a}(t) = \langle 0, -g \rangle$

$\Longrightarrow \bar{v}(t) = \langle 0, -gt \rangle + \bar{C}_1 \quad \text{---} (1)$

Let the muzzle speed be V_0. Then $\bar{V}(0) = \langle V_0 \cos\frac{\pi}{3}, V_0 \sin\frac{\pi}{3} \rangle$.

$\Longrightarrow \bar{V}(0) = \langle \frac{V_0}{2}, V_0\frac{\sqrt{3}}{2} \rangle$

$\therefore (1) \Longrightarrow \bar{V}(t) = \langle \frac{V_0}{2}, V_0\frac{\sqrt{3}}{2} - gt \rangle$

$\Longrightarrow \bar{r}(t) = \langle \frac{V_0 t}{2}, V_0\frac{\sqrt{3}}{2}t - g\frac{t^2}{2} \rangle + \bar{C}$

Since $\bar{r}(0) = \langle 0, 0 \rangle$, $\bar{r}(t) = \langle \frac{V_0 t}{2}, V_0\frac{\sqrt{3}}{2}t - g\frac{t^2}{2} \rangle$ //

Let t_1 be the time when the shell strikes the target.
Then the y-comp of $\bar{r}(t_1) = 0$, and x-comp $= 6000$.

$\Longrightarrow V_0\frac{\sqrt{3}}{2}t_1 - g\frac{t_1^2}{2} = 0 \Longrightarrow \frac{t_1}{2}(V_0\sqrt{3} - gt_1) = 0.$

$\Longrightarrow t_1 = 0$ or $t_1 = \frac{V_0\sqrt{3}}{g}$

Since $\frac{V_0 t_1}{2} = 6000$, $\frac{V_0}{2}\left(V_0\frac{\sqrt{3}}{g}\right) = 6000.$

$\Longrightarrow V_0^2 = \frac{12000g}{\sqrt{3}} \Longrightarrow V_0 = \sqrt{\frac{12000g}{\sqrt{3}}} \approx 261 \text{ m/s}.$ //

(6) Assume that $z = f(x, y)$ in the equation $3x^2z^2 - xyz + y^2z^3 = 3$. Find the slope of z, in the $x-$ direction at $(1, 1, 1)$.

Show all your reasoning, in a step by step manner, which leads to the logical conclusion.

Implicitly diff both sides wrt x, holding y const.

$$\Longrightarrow \quad 6xz^2 + 3x^2 \cdot 2z\frac{\partial z}{\partial x} - yz - xy\frac{\partial z}{\partial x} + 3y^2z^2\frac{\partial z}{\partial x} = 0$$

$$\Longrightarrow \quad \frac{\partial z}{\partial x} = \frac{6xz^2 - yz}{xy - 6x^2z - 3y^2z^2}$$

$$\Longrightarrow \quad \left.\frac{\partial z}{\partial x}\right|_{(1,1,1)} = \frac{6-1}{1-6-3}$$

$$= -\frac{5}{8}$$

(7) (a) Write the definition of the differentiability of $z = f(x, y)$, at a given point (x_0, y_0) precisely.
(b) Write the equation of the local linear approximation to $z = f(x, y)$ at the point (x_0, y_0) precisely.
(c) Find the local linear approximation to $z = x^2y - x^2 - y^2$ at the point $(1, 1)$.

(a) We say $z = f(x, y)$ is differentiable at (x_0, y_0) if $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ exist and

$$\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y) - f(x_0,y_0) - f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0.$$

//

(b) $L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$ //.

(c) $\frac{\partial z}{\partial x} = 2xy - 2x \Longrightarrow \frac{\partial z}{\partial x}(1,1) = 0.$

$\frac{\partial z}{\partial y} = x^2 - 2y \Longrightarrow \frac{\partial z}{\partial y}(1,1) = -1$

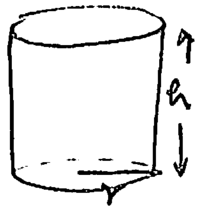
$z(1,1) = -1.$

$\therefore$ $L(x,y)$ to $f(x,y)$ at $(1,1)$ is:

$L(x,y) = -1 - 1(y-1)$

$\Longrightarrow L(x,y) = -y$ //

(8) The radius and the height of a right circular cylinder are measured with errors of at most 0.1 inches. If the height and the radius are measured to be 10 inches and 2 inches, respectively, use differentials to approximate the maximum possible error in the calculated value of the volume. (Volume of a right circular cylinder is $\pi r^2 h$, where r is the radius and h is the height.)
Show all your reasoning, in a step by step manner, which leads to the logical conclusion.



$$r \pm \Delta r = 2 \pm .1 \text{ in}$$
$$h \pm \Delta h = 10 \pm .1 \text{ in}$$

$$V = \pi r^2 h$$

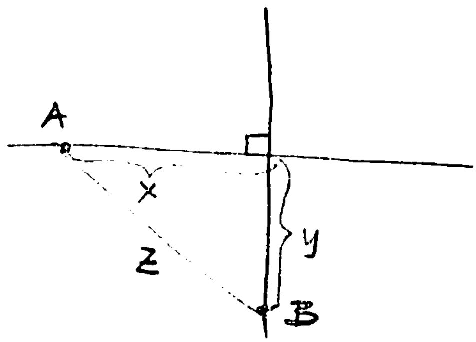
$$\Longrightarrow dV = 2\pi r h \, dr + \pi r^2 dh.$$

Let $dr = \Delta r$, $dh = \Delta h$. Then $dV \approx \Delta V$,

$$\Longrightarrow |\Delta V| \approx |2\pi r h \Delta r + \pi r^2 \Delta h|$$
$$\leqslant 2\pi r h |\Delta r| + \pi r^2 |\Delta h|, \text{ by triangle inequality.}$$
$$= 2\pi(2)(10)(.1) + \pi(2)^2(.1)$$
$$= .4\pi(10+1)$$
$$= 4.4\pi \approx 13.8 \text{ in}^3$$

$\therefore$ The maximum possible error in the calculated value of V is 13.8 in^3.

(9) Two straight roads intersect at right angles. Car A, moving on one of the roads, approaches the intersection at 35 mi/h and car B, moving on the other road, approaches the intersection at 40 mi/h. At what rate is the distance between the cars changing when A is 0.3 mile from the intersection and B is 0.4 mile from the intersection? Show all your reasoning, in a step by step manner, which leads to the logical conclusion.



Let the dist. between Car A and the intersection be $x(t)$ mi, the dist. between car B and the intersection be $y(t)$ mi, and the dist. between Car A and Car B be $z(t)$ mi.

$$\Rightarrow \quad z(t) = \sqrt{x^2 + y^2}$$

$$\Rightarrow \quad \frac{dz}{dt} = \frac{x}{\sqrt{x^2+y^2}} \frac{dx}{dt} + \frac{y}{\sqrt{x^2+y^2}} \frac{dy}{dt}$$

$$\Rightarrow \quad \frac{dz}{dt}(x=0.3,\ y=0.4) = \frac{0.3}{0.5}(-35) + \frac{0.4}{0.5}(-40)$$

$$= -53 \text{ mi/h.}$$

$\therefore$ The dist. between the cars is decreasing at a rate of 53 mi/h. //

(10) Does $\lim_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x^2+y^2}$ exist? If the answer is yes then find the limit. If the answer is no then justify your answer by carefully choosing and finding the limit along several smooth curves that passes through $(0,0)$.
Show all your reasoning, in a step by step manner, which leads to the logical conclusion.

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along x-axis}}} \frac{x^2-y^2}{x^2+y^2} = \lim_{t\to 0} \frac{t^2-0}{t^2+0} = 1.$$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along y-axis}}} \frac{x^2-y^2}{x^2+y^2} = \lim_{t\to 0} \frac{0-t^2}{0+t^2} = -1.$$

$$\therefore \lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x^2+y^2} \text{ does not exist.} //$$