MTH 202: Exam 3 **Name:** key
Time: 75 minutes

Use of notes, note cards, cellphones or PDAs are not allowed during the quiz.

(1) Find the equations of the tangent plane and the normal line to $4x^2 + 9y^2 + z = 17$ at $(-1, 1, 4)$.
Show all your reasoning, in a step by step manner, which leads to the logical conclusion.

$z = 17 - 4x^2 - 9y^2$

$\Rightarrow \frac{\partial z}{\partial x} = -8x$ and $\frac{\partial z}{\partial x}(-1,1,4) = 8$.

$\frac{\partial z}{\partial y} = -18y$ and $\frac{\partial z}{\partial x}(-1,1,4) = -18$

$\therefore$ The equation of the tangent plane at $(-1,1,4)$ is:

$8(x+1) - 18(y-1) - (z-4) = 0$ OR

$8x - 18y - z + 30 = 0$ //

The parametric equations of the normal line are:

$$\left.\begin{aligned} x(t) &= -1 + 8t \\ y(t) &= 1 - 18t \\ z(t) &= 4 - t \end{aligned}\right\} \text{ for some parameter } t. //$$

(2) Locate all relative maxima, relative minima and saddle points for $f(x, y) = x^3 - 9xy + y^3$.

Show all your reasoning, in a step by step manner, which leads to the logical conclusion.

$f_x = 3x^2 - 9y$

$f_y = -9x + 3y^2$

$f_x = 0$ <u>and</u> $f_y = 0 \Rightarrow \quad x^2 - 3y = 0$ — ①

$-3x + y^2 = 0$ — ②

① $\Rightarrow y = \frac{x^2}{3}$. Substitute in ②:

$-3x + \frac{x^4}{9} = 0 \Rightarrow x(x^3 - 27) = 0$

$\Rightarrow x = 0$ or $x = 3$

$\Rightarrow (0,0)$ and $(3,3)$ are critical pts.

$f_{xx} = 6x, \ f_{xy} = -9, \ f_{yx} = -9, \ f_{yy} = 6y$

$\Rightarrow D(x,y) = 36xy - 81$

<u>At $(0,0)$</u>: $f_{xx}(0,0) = 0, \ D(0,0) = -81 < 0$.

$\therefore$ There is a saddle pt at $(0,0)$. //

<u>At $(3,3)$</u>: $f_{xx}(3,3) = 18 > 0, \ D(3,3) = 36 \cdot 9 - 81 = 9(36-9) > 0$,

$\therefore$ There is a rel. min. at $(3,3)$ //

(3) Find the maximum sum of $x^2 + y^2 + z^2$ if $x + 2y + 2z = 12$.
Show all your reasoning, in a step by step manner, which leads to the logical conclusion.

Let: $f(x,y,z) = x^2+y^2+z^2$ and $g(x,y,z) = x+2y+2z-12$

$\Rightarrow \overline{\nabla} f(x,y,z) = \langle 2x, 2y, 2z \rangle$ & $\overline{\nabla} g(x,y,z) = \langle 1, 2, 2 \rangle$.

$\overline{\nabla} f = \lambda \overline{\nabla} g \Longrightarrow$
$2x = \lambda$ — ①
$2y = 2\lambda$ — ②
$2z = 2\lambda$ — ③

& the constraint, $x+2y+2z = 12$ — ④

Substitute $x = \frac{\lambda}{2}$, $y = \lambda$, $z = \lambda$ in ④;

$\Longrightarrow \frac{\lambda}{2} + 2\lambda + 2\lambda = 12 \Longrightarrow 9\lambda = 24 \Longrightarrow \lambda = \frac{8}{3}$.

$\therefore x = \frac{4}{3}$, $y = \frac{8}{3}$, $z = \frac{8}{3}$.

$\Rightarrow f(\frac{4}{3}, \frac{8}{3}, \frac{8}{3}) = \frac{16}{9} + \frac{64}{9} + \frac{64}{9} = \frac{144}{9} = 16$.

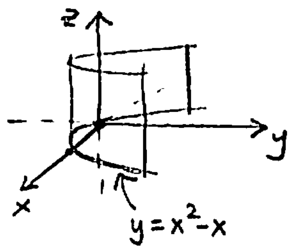
$(0,0,6)$ is a point on the plane $x+2y+2z=12$.

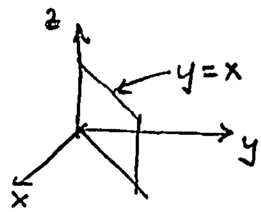
$\therefore f(0,0,6) = 24 > 16$

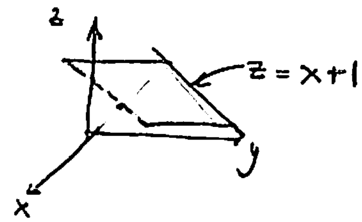
$\Rightarrow$ f has a constraint rel. min at $(\frac{4}{3}, \frac{8}{3}, \frac{8}{3})$ and the const. rel. min. is 16. //

There is <u>no</u> const. rel. max. //

(4) Find the volume of the solid enclosed by $y = x^2 - x$, $y = x$ and $z = x + 1$. Do not forget to sketch the solid and the region. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.





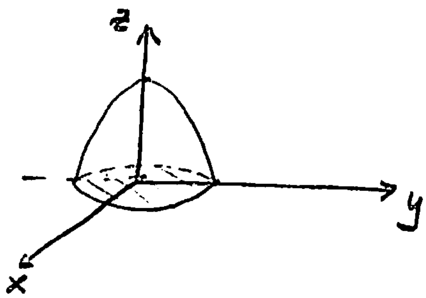


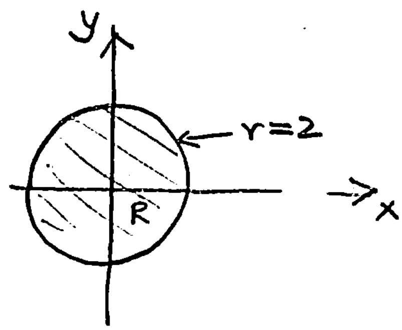
This solid is not bdd. //

i.e The volume is ∞,

(5) Find the volume of the solid enclosed by the paraboloid $z = 4 - x^2 - y^2$ and $z = 0$.

Do not forget to sketch the solid and the region. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.





$$\Longrightarrow \iint_R 4 - x^2 - y^2 \, dA = \int_0^{2\pi}\int_0^2 (4 - r^2)\, r\, dr\, d\theta$$

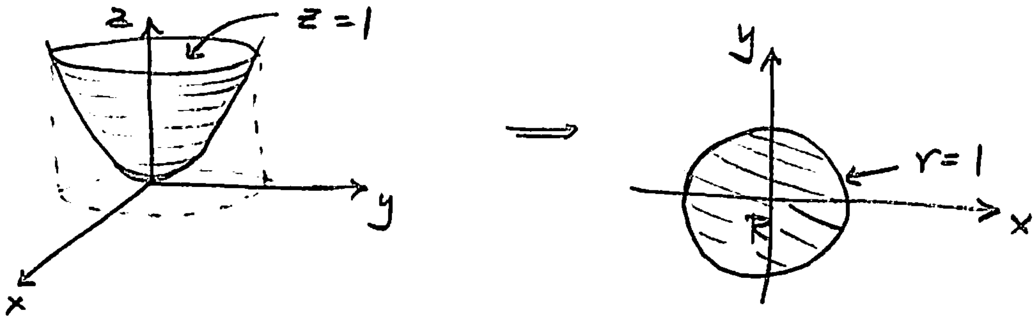
$$= \int_0^{2\pi} 2r^2 - \frac{r^4}{4} \Big|_0^2 \, d\theta$$

$$= \left(8 - \frac{16}{4}\right) 2\pi$$

$$= \underline{\underline{8\pi}} \text{ vol. units,}$$

(6) Find the surface area of the portion of the paraboloid $z = x^2 + y^2$ which lies below the plane $z = 1$.

Do not forget to sketch the surface and the region. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.



$$\frac{\partial z}{\partial x} = 2x, \quad \frac{\partial z}{\partial y} = 2y$$

$$\Longrightarrow \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2 + 1 = 4x^2 + 4y^2 + 1 = 4(x^2 + y^2) + 1$$

$$\Longrightarrow \text{Surface Area} = \int_0^{2\pi} \int_0^1 \sqrt{4r^2 + 1}\, r\, dr\, d\theta$$

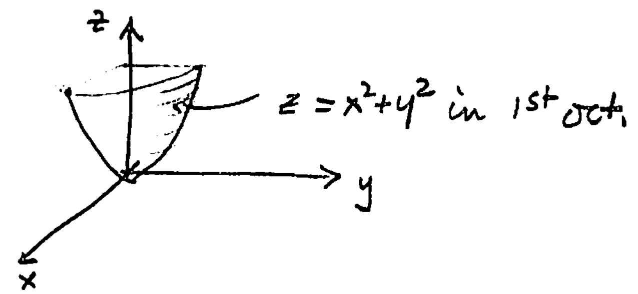
$$= \frac{1}{8} \cdot \frac{2}{3} \int_0^{2\pi} (4r^2 + 1)^{3/2} \Big|_0^1 \, d\theta$$

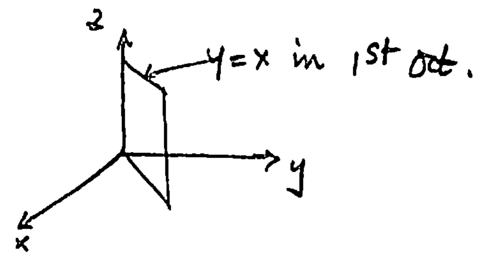
$$= \frac{1}{12} (5^{3/2} - 1) 2\pi$$

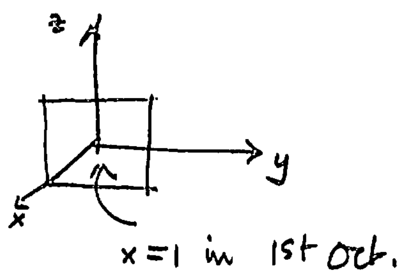
$$= \frac{1}{6} (5\sqrt{5} - 1) \pi \text{ square units.} //$$

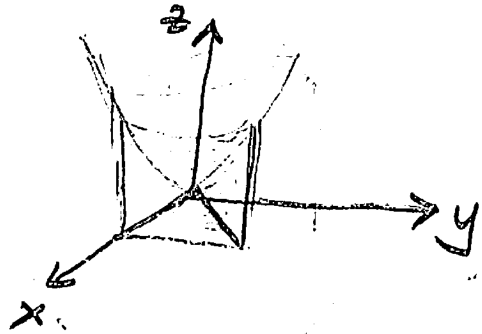
(7) Find the volume of the solid in the first octant enclosed by $z = x^2 + y^2$, $y = x$ and $x = 1$.

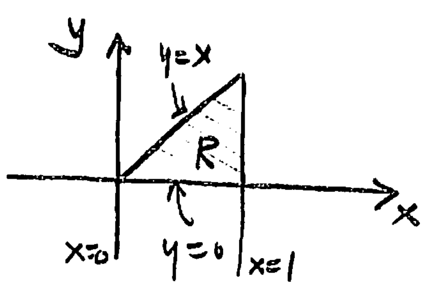
Do not forget to sketch the solid and the region. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.









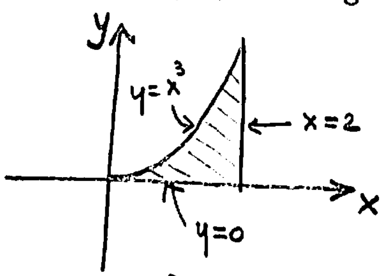


$$Vol = \int_0^1 \int_0^x \int_0^{x^2+y^2} dz\, dy\, dx$$

$$= \int_0^1 \int_0^x x^2+y^2\, dy\, dx = \int_0^1 x^2 y + \frac{y^3}{3} \Big|_0^x dx$$

$$= \int_0^1 x^3 + \frac{x^3}{3} dx = \frac{4}{3} \int_0^1 x^3 dx = \frac{4}{3} \frac{x^4}{4} \Big|_0^1 = \frac{1}{3} \text{ V.u.}$$

(8) Find the centroid of the lamina enclosed by $y = x^3$, $x = 2$ and $y = 0$.
Do not forget to sketch the region. Show all your reasoning, in a step by step manner, which leads to the logical conclusion.



Type I region.

$$\int_0^2\int_0^{x^3} x\,dy\,dx = \int_0^2 xy\Big|_0^{x^3}dx = \int_0^2 x^4\,dx = \frac{x^5}{5}\Big|_0^2 = \frac{32}{5}.$$

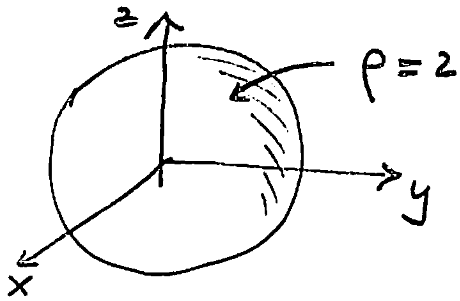
$$\int_0^2\int_0^{x^3} y\,dy\,dx = \frac{1}{2}\int_0^2 y^2\Big|_0^{x^3}dx = \frac{1}{2}\int_0^2 x^6\,dx = \frac{1}{14}x^7\Big|_0^2 = \frac{64}{7},$$

$$\int_0^2\int_0^{x^3} dy\,dx = \int_0^2 x^3\,dx = \frac{x^4}{4}\Big|_0^2 = 4.$$

$\therefore$ centroid $= \left(\frac{8}{5}, \frac{16}{7}\right)$

(9) Find the mass of the sphere $x^2 + y^2 + z^2 = 4$ if its density is given by $\delta(x, y, z) = x^2 + y^2$.

Show all your reasoning, in a step by step manner, which leads to the logical conclusion.



$$\Longrightarrow \text{Mass of the sphere} = \iiint_G (x^2 + y^2)\, dV$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{2} \left(\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta\right) \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta$$

$$= \int_0^{2\pi}\int_0^{\pi}\int_0^{2} \rho^4 \sin^3\phi \, d\rho\, d\phi\, d\theta$$

$$= \frac{32}{5}\int_0^{2\pi}\int_0^{\pi} \sin^3\phi \, d\phi\, d\theta$$

$$= -\frac{32}{5}\int_0^{2\pi}\int_0^{\pi} -(1 - \cos^2\phi)\sin\phi \, d\phi\, d\theta$$

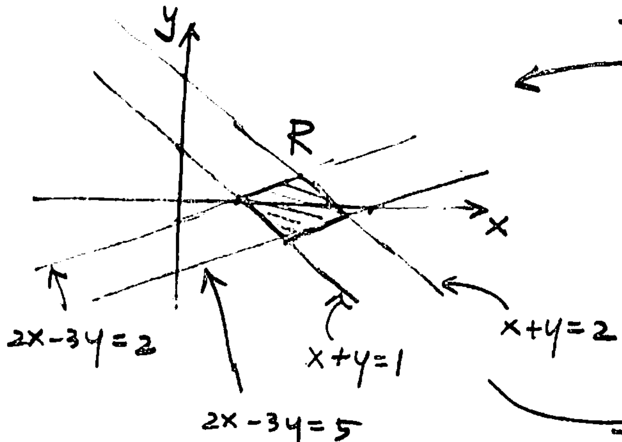
$$= -\frac{32}{5}\int_0^{2\pi} \cos\phi - \frac{\cos^3\phi}{3}\Big|_0^{\pi} d\theta$$

$$= -\frac{32}{5}\left(\left(-1 + \frac{1}{3}\right) - \left(1 - \frac{1}{3}\right)\right)\int_0^{2\pi} d\theta$$

$$= \frac{32}{5} \cdot \frac{4}{3} \cdot 2\pi = \frac{256}{5}\pi \text{ vol. units}$$

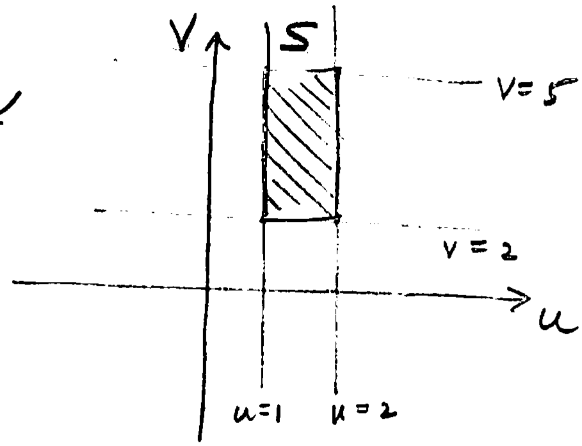
(10) Evaluate $\displaystyle\iint_R x\,dA$ where R is the region bounded by $x+y=1$, $x+y=2$, $2x-3y=2$, $2x-3y=5$.

Show all your reasoning, in a step by step manner, which leads to the logical conclusion.



$T:\ x = \dfrac{3u+v}{5}$, $\quad y = \dfrac{2u-v}{5}$

$T^{-1}:\ u = x+y$, $\quad v = 2x-3y$



$$J(x,y) = \begin{vmatrix} \frac{3}{5} & \frac{1}{5} \\ \frac{2}{5} & -\frac{1}{5} \end{vmatrix} = -\frac{3}{25} - \frac{2}{25} = -\frac{1}{5}$$

$$\therefore \iint_R x\,dA_{xy} = \iint_S \frac{3u+v}{5}\,dA_{uv}$$

$$= \frac{1}{5}\cdot\frac{1}{5}\int_2^5\int_1^2 (3u+v)\,du\,dv$$

$$= \frac{1}{25}\int_2^5 \left.\frac{3u^2}{2}+uv\right|_1^2 dv = \frac{1}{25}\int_2^5 (6+2v)-\left(\frac{3}{2}+v\right)dv$$

$$= \frac{1}{25}\int_2^5 \left(\frac{9}{2}+v\right)dv = \frac{1}{25}\left[\frac{9}{2}v+\frac{v^2}{2}\right]_2^5 = \frac{1}{50}\left[(45+25)-(18+4)\right]$$

$$= \frac{24}{25}$$